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Construction and application of a mechanical differential analyzer

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CONSTRUCTION AND APPLICATION OF A MECHANICAL DIFFERENTIAL ANALYZER

by

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INTRODUCTION

In the course of a search for a convenient and inexpensive means of solving trajectory equations and certain other problems associated with the balloon assisted rocket flights being made in connection with cosmic ray research by the Physics Department of the State University of Iowa, it was discovered that the construction of a versatile, adequately precise, and quite inexpensive small modified version of the Bush differential analyzer was a feasible idea, that such a machine would be able to solve the particular problems mentioned, and that the further great advantage would be gained that such a computer could be applied to a wide range of other problems as well.

The computer has been built and used, and part of the present paper is devoted to a description of the machine, an account of the manner in which it was tested, and an evaluation of the precision attainable by it under various operating conditions.

Another part of this paper is then given over to a discussion of the rocket problems and to the solutions obtained by use of the computer.

Also incorporated in this paper, and forming much of the bulk of it, is a complete manual of operation for the machine. This part of the paper was written with a reader in mind who was assumed to be completely ignorant of analog computers. Accordingly, an attempt was made to give so thorough a discussion of all points involved that such a reader could, by a reading of the pertinent parts of this paper, be made capable of using the machine effectively at once.

It ought to be pointed out here that neither the idea of the differential analyzer, nor the idea of a small version of it, is new. The invention itself was originally due to Lord Kelvin,[1] although the first successful operating machine was made by Bush[2] and his co-workers. The idea of a small version apparently springs from Hartree,[3] and while it is not known whether any such machines capable of serious scientific applications exist in this country other than the S.U.I. machine, it is known

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1. Thomson, Sir W. (Lord Kelvin) Proc. Roy. Soc. 24 (1876), 269.
2. Bush, V., J. Frank. Inst. 212 (1931), 447.
3. Hartree, D. R., and Porter, A., Mem. and Proc. Manch. Lit. and Phil. Soc., 79, (1935), 51.

that at least four have been made and used in England.[4,5,6,7]

The State University of Iowa machine is of a new design, specially adapted to the purpose in view, to the materials available without purchase, to economy in purchased parts, to ease in carrying out the machine work required, and incorporates some new ideas that very materially improve on previous machines of its type.

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4. Ibid.
5. Massey, H. S. W., Wylie, I., Buckingham, R. A., and Sullivan, R., Proc. Roy. Irish Acad. 45a (1938), 1.
6. Beard, R. E. Roy. Coll. Sci. J. 12 (1942), 99.
7. Wood, A. M. M. Sc. Thesis, University of Birmingham, 1942.

Chapter I

THEORY AND GENERAL DESCRIPTION

(1) The General Idea of Analog Computers

It has been shown by Miller[8] that any of an extremely wide class of ordinary differential equations can be solved by an ensemble of units of certain special natures. It is proposed to show first how such an ensemble can be used to effect the solutions, without specifying exactly how the units might be realized physically. After this, the manner in which the required units can be constructed in mechanical form will be described.

The units referred to above are:

(a) Integrators, capable of integrating continuously any input function of one variable with respect to any independent variable:

(b) Adders, capable of producing the sum or difference of any quantities;

(c) Multipliers, able to form the instantaneous product of two input functions;

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8. Miller, F. J., Theory of Mathematical Machines, King's Crown Press, Morningside Heights, New York, 1947, p. 81.

(d) Constant factor devices, able to provide an output which is a preselected constant times the input function;

(e) Function generators, which are units that must be able to accept as input any variable of interest, and to provide as output any arbitrary function of that variable.

Not included in this list, but to be understood as being required for practical reasons, are auxiliary devices for coupling the various units into a network, for supplying driving power to the machine, and for recording the solution.

It will now be assumed that units of these five kinds are available, along with all necessary auxiliary devices. (The discussion that follows will apply equally as well to electronic analog computers as to mechanical ones, and it will only be when the exact nature of the units is specified that the generality of the discussion will be reduced.)

The manner in which the assembly of units is used in the solution of a differential equation can be most easily made clear by a simple example; the idea is easily grasped, and the cause of brevity is served by illustrating

it rather than by describing it at length. The example chosen for use here is the non-linear equation $A\ddot{x} + f(x)\dot{x} + Bx = 0$. Here the dots indicate the derivatives, A and B are constants, and f(x) is an arbitrary function which will not need to be specified explicitly.

Reference is now made to Figures 1 and 2. Figure 2 comprises an explanation of the symbols used in Figure 1, and this latter figure is itself a schematic block diagram representation of one way in which the solution of the equation given above could be arranged. In this circuit, any point can be taken as beginning point, but it is most convenient to begin at the upper left, where a line is indicated as being a physical representation, such as a voltage or shaft rotation, of the quantity $A\ddot{x}$.

Units 1 and 2 perform integrations with respect to t, and the factors C and D that appear are shown to indicate that integrators invariably produce the integral with a (known) "integrator factor" before it. Unit 4 takes in the quantity Dx and puts out f(x), which goes into the multiplier, unit 5, along with the quantity $\dot{x}$ which has been produced from the quantity $C\dot{x}$ by unit 3. The output of unit 5 is the product $-f(x)\dot{x}$, where the negative sign has been supplied within the unit; a possible factor before the product has been omitted solely for simplicity,

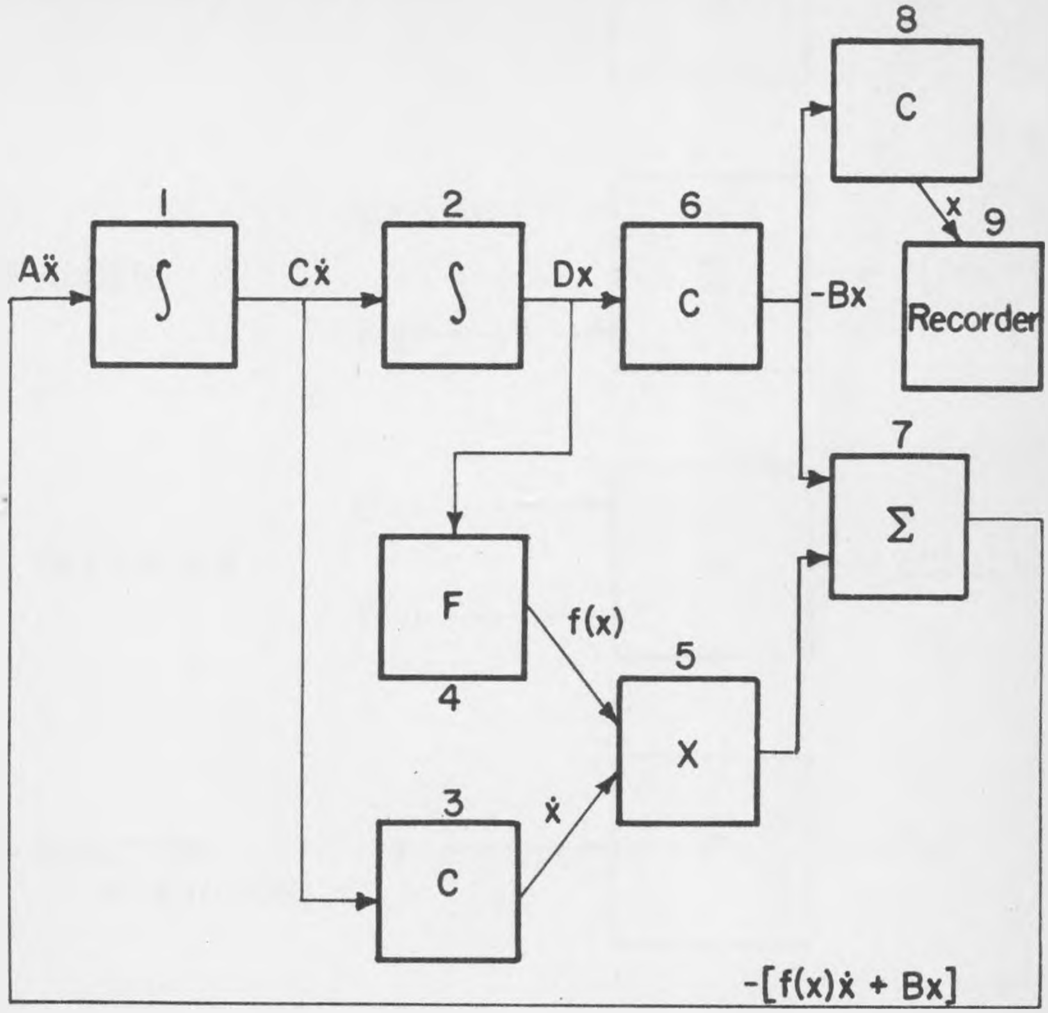
1
2
6
8
9
7
5
4
3
$A\ddot{x}$
$\int$
$C\dot{x}$
$\int$
Dx
C
C
x
Recorder
$-Bx$
Σ
F
$f(x)$
X
$\dot{x}$
C
$-[f(x)\dot{x} + Bx]$

FIG. 1.

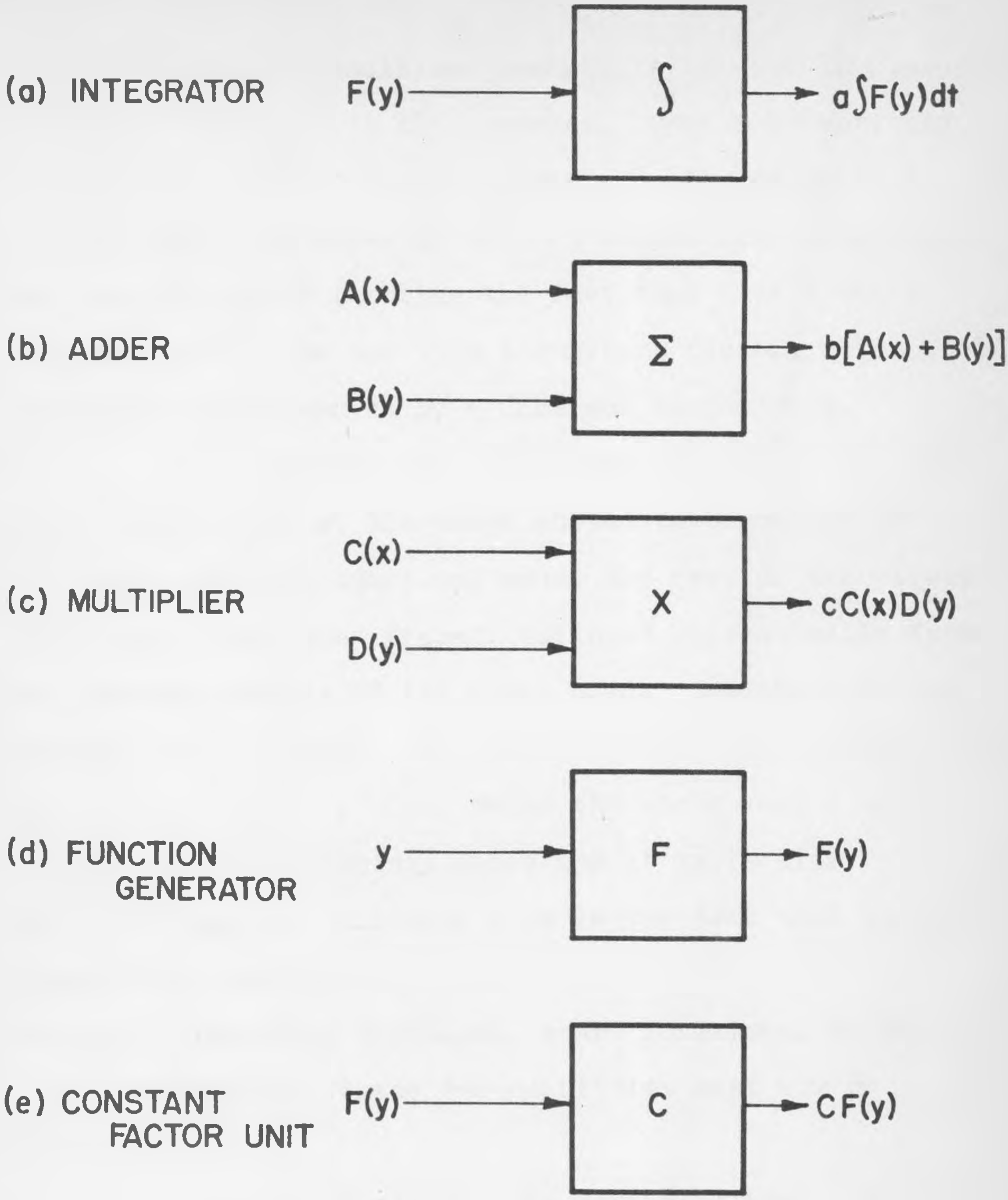
(a) INTEGRATOR
F(y)
∫
a∫F(y)dt
(b) ADDER
A(x)
B(y)
Σ
b[A(x) + B(y)]
(c) MULTIPLIER
C(x)
D(y)
X
cC(x)D(y)
(d) FUNCTION GENERATOR
y
F
F(y)
(e) CONSTANT FACTOR UNIT
F(y)
C
CF(y)

FIG. 2.

since including it would not contribute to ease in grasping the scheme involved in this problem. Unit 6 effectively changes the factor D before x into -B. The outputs of units 5 and 6 are added by unit 7, where again simplicity has been sought by ignoring the fact that unit 7 would actually supply the sum with a constant factor, which would then have to be removed by a Constant Factor Unit.

It is to be noted that tracing the circuit was begun arbitrarily at the point chosen to be called $A\ddot{x}$, and that once this start was made, the rest of the circuit, as it has so far been traced, followed automatically from the assumed natures of the units used. Now the most important step is taken: the output of unit 7 is united with the input of unit 1. This makes the whole mesh a closed circuit, and the original loose end at $A\ddot{x}$ is eliminated. The importance of this step lies in the fact that it enforces the equality of $A\ddot{x}$ and $-(f(x)\dot{x} + Bx)$, since the physical quantities (voltages, shaft rotations, or the like) representing these two quantities must now be identical.

The closed circuit, as just mentioned, exemplifies, or realizes, the given equation in analog form, and is furthermore a dynamic system -- although the discussion in the next paragraph must be taken into account in this

connection -- so that it must change its state in time. As it does so, the quantities $\ddot{x}$, $\dot{x}$, and x appearing in it must be those quantities $\ddot{x}$, $\dot{x}$, and x which appear in the original equation, aside from scale factors which do not affect the sense of this statement. If, for example, the solution x is wanted, it can be tapped out at the output of unit 6, and recorded in graphical form by a recording unit, shown as unit 9, after a possible scale change brought about by unit 8.

Actually, two points concerning the dynamic character of the system were neglected in the above discussion, for the sake of initial simplicity. The first is that this character is really due to external sources of driving power not shown in the figure; for example, in a mechanical computer an auxiliary motor drives the integrators and accounts for the development of the state of the system in time. The second point is that the circuit as described above would not actually be a dynamic system until means are provided for starting the various quantities involved off with their proper initial values. This is clear from the fact that if x and all its derivatives are initially zero, the mesh must stay in a state of quiescence. Any actual analog machine must contain means of supplying appropriate initial conditions for the

variables, but there is no perfectly general way of doing this. In electronic analog computers, D. C. supplies are used to make the voltages representing x and its derivatives have the proper values at the beginning instant. The means for accomplishing this end in the mechanical differential analyzer will be described later.

By way of summary, the general way in which a differential analyzer, whether of the mechanical or electronic variety, is used to solve an equation can be described as follows: One has available in the machine a number of units of Miller's five types, which one interconnects in such a way that the relations among the machine variables are the same as those imposed on the corresponding mathematical variables by the given equation.

It is well to anticipate here a fact that will be discussed at length later -- namely, that the planning of a machine set-up in any given case in full detail, including consideration of all scale factors, constant values, and other such matters, is by no means as simple as the drawing of a block diagram of the type shown in Figure 1. Indeed, a genuing programming procedure of some complexity must be gone through, analogous in many ways to the programming of a digital computer.

(2) The Mechanical Differential Analyzer

The mechanical differential analyzer is a collection of devices that are translations into mechanical terms of the operating units referred to in Section (1), and of such auxiliary devices as the inter-unit connections. In the present section, the manner in which mechanical devices can be made to perform integrations, additions, multiplications, and function generation will be described, and the general method of linking them together into a functioning whole will be outlined.

In the first place, all mathematical variables become, in the machine, new variables, which are physical rotations of shafts. The relations between the variables are given by means of scale factors, of the nature of shaft rotations per unit of variable; thus, when a mathematical variable has a value x, the number of rotations of that shaft which represents the variable will be X, related to x by $X = Rx$, R being the factor in question.

The factor that changes any one variable into rotations is not generally that which changes another variable into rotations, nor for any one variable is the factor the same in all parts of the machine where that variable appears. However, the relations between the

variables and their associated factors enforced by the machine set-up must always be consistent with the relations fixed by the differential equation being solved.

(a) Integrating Units

The integrators constitute the fundamental units in the differential analyzer. Mechanical integrators of various kinds exist, but only one type will be described here: this is the device known as a Kelvin disc integrator. In Figure 3a is given a representation of such an integrator. Here a is a horizontal circular disc mounted rigidly on the vertical axis d, which is geared through a miter-gear pair to shaft e within the carriage c. In this way, a rotation of shaft e communicates a corresponding rotation to the disc a. There is a small wheel b which rests of its own weight on disc a and follows the rotations of the disc without slipping. Shaft f is rigidly attached to wheel b, and the shaft and wheel are held fixed with respect to translational motion. The disc a has one degree of translational freedom under the wheel b. Lateral motions of the disc under the wheel are brought about by rotations of the lead screw g, which turns in a nut attached to the carriage c. The carriage runs on wheels on rails, or slides in tracks, so that the lateral motion of the

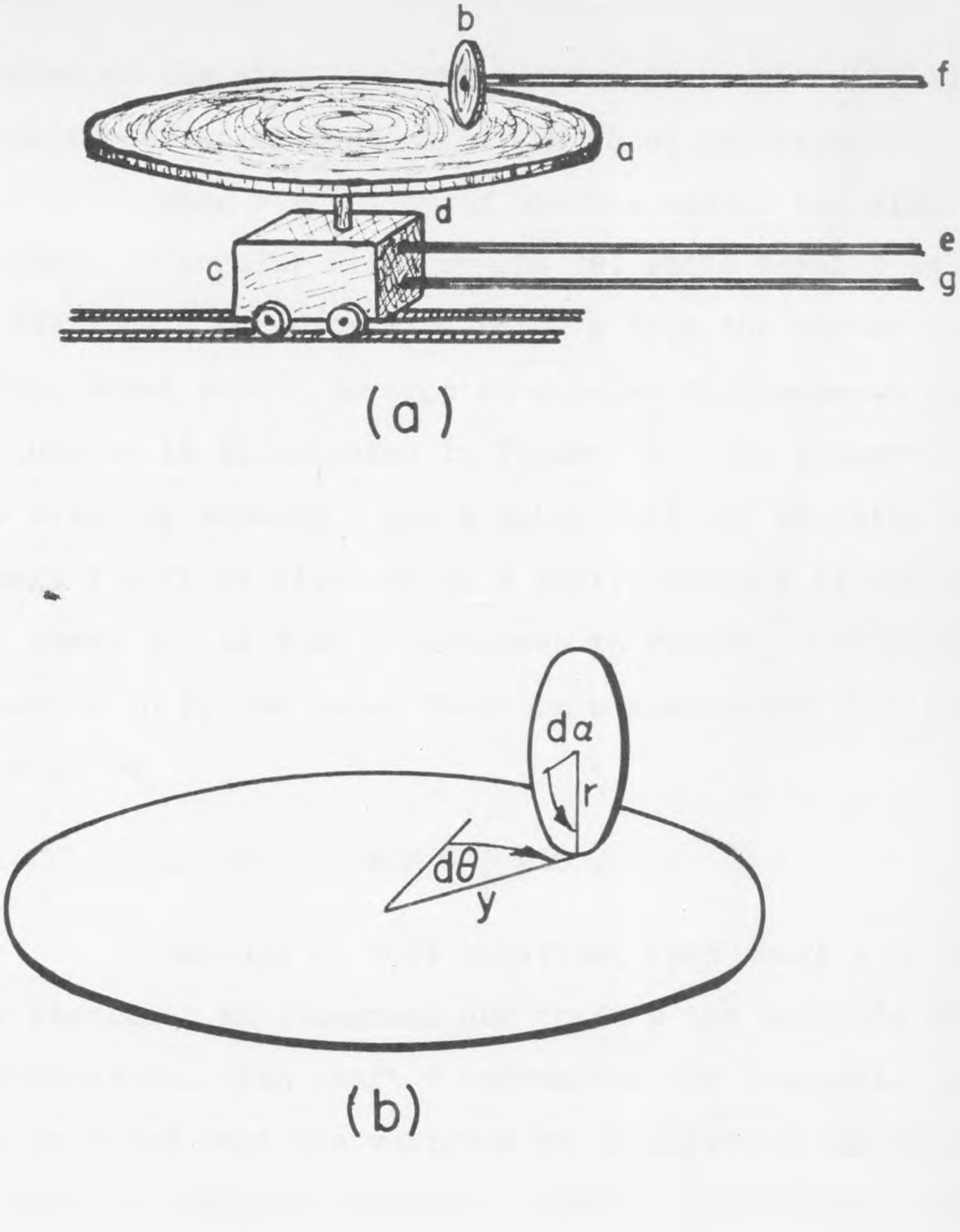

FIG. 3.

center of the disc is along a fixed line which includes always the point of contact of the wheel and disc.

When a rotation of shaft e causes the disc a to undergo an angular displacement $d\theta$, while wheel b is at a distance y turns of lead screw g from the center position, wheel b will undergo an angular displacement $d\alpha$. This situation is illustrated in Figure 3b. The assumption of no slipping between a and b means that the rotation of shaft f will be given by $d\alpha = y d\theta / r$, where r is the radius of wheel b. As disc a continues to rotate, with concurrent changes in y, the total rotation α associated with shaft f will be

$$\alpha = \frac{1}{r} \int y d\theta .$$

Because of this relation, when shaft g is made to represent an integrand and shaft e the variable of integration, then shaft f represents the integral. It is to be noted that the variable of integration can be any variable, and the integrand a function of any variable. Also, such a mechanical integrator is not limited to integrations with respect to real time or a multiple thereof, as are the amplifiers used as integrators in electronic analog computers. Indeed, essentially because the integrators can be run slowly, rapidly, or even at varying speeds

during a solution, there is no immediate necessary relation at all between the passage of real time and progress of the solution.

The relation given above, expressing α, needs to be modified when any actual integrator unit is referred to, for the reasons that the variable of integration shaft is usually geared down before being allowed to turn the disc, the output, or integral, shaft is also geared down before being interconnected with other parts of the machine, and the lead screw that drives the carriage will have a screw pitch that requires it to turn a certain number of times to make a unit displacement of the carriage. Calling the first gear ratio K, the second L, and the pitch of the lead screw P, one easily sees that the rotation of the output shaft from an integrator is given by

$$\alpha = \frac{KLP}{r} \int y d\theta.$$

The quantity KLP/r is called the integrator constant and is always made to be the reciprocal of an integer, with the integer in question a power of two, for the sake of convenience. As an example, in the State University of Iowa machine, for each integrator P is 20, K is unity, L is 1/2, and the wheel diameter is 3.200 $\pm$ .002 inches; with these values the constant reduces to

1/64. The wheel diameter was preselected to give this value, and was achieved by very careful grinding of a steel disc.

(b) Adders

An adding unit, in mechanical computers, is an arrangement of gears that interconnects three shafts in such a way that the algebraic sum of the rotations of two of them appears as the rotation of the third. This arrangement of gears is essentially a differential gear. Figure 4 shows an adder; there are four bevel gears, of which a and b are fixed to the shafts A and B respectively, while c and d are free of the shaft D. Shaft D is rigidly attached to the metal casing that contains the whole. While this casing is completely free of all shafts, it rotates about the axis formed by the shafts A and B. S_1 is a spur gear attached to the casing, and communicates the motion of the casing through spur gear S_2 to shaft C. (It will be assumed that the gears S_1 and S_2 are in the ratio 1:1.)

The operation of the unit can be understood in the following way: Consider first that B is stationary while A rotates with angular velocity α. Calling the radius of each bevel gear r, it is clear that the common

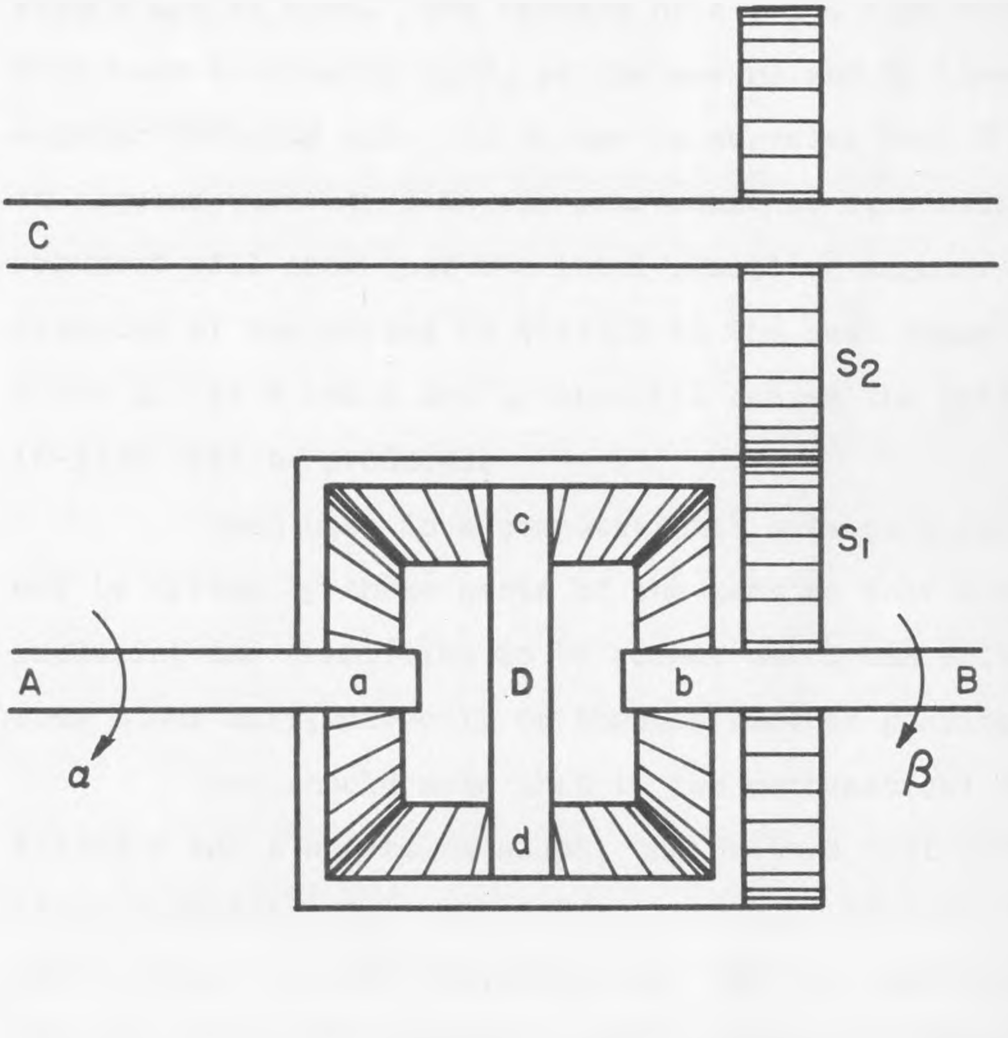

FIG. 4.

velocity of the points where a and c, and also a and d, are in contact, is $r\alpha$. The points of c and d in contact with b are at rest. The centers of c and d must therefore have a velocity $r\alpha/2$, so the casing and S_1 have angular velocity $\alpha/2$. If it now be supposed that B has an angular velocity β in the same sense as α, similar argument will show that the total resulting angular velocity of the casing is $(\alpha+\beta)/2$ in the same sense as α and β. If α and β are in opposite senses the difference $(\alpha-\beta)/2$ will be produced.

When used in a computational set-up, A and B may be driven by those parts of the machine that are producing the quantities to be added, and C can drive some other unit, directly or through further gearing.

One should note that if two mathematical quantities f and F are to be added, the factors that convert these quantities into shaft rotations must be made the same before the addition; otherwise the two quantities will be added with unequal weights attached to them, depending on the factors involved. In order that this be quite clear, suppose that the machine quantities Af and BF are available in the machine, and that the sum of f and F is to be produced, along with a possible change of scale factor. Then A must be converted into B, or

B into A, before the adder is used; the adder output will then be $A(f + F)/2$ in the first case, or $B(f + F)/2$ in the second case.

(c) Multipliers

For the sake of brevity in the present discussion, only the general notion underlying the operation of mechanical multiplying units will be given. In the MIT, Cambridge, and other differential analyzers, actual constructional details of the multipliers used vary quite widely, and the multiplier in the State University of Iowa machine is still different. Thus description of details is best postponed, while only the underlying idea need be given here; this idea is basically the same in all cases.

In its simplest form, the multiplying bar forms the main part of the unit. This is a metal bar in the form of a right angle, lying parallel to a plane board, and free to rotate about a pivot, marked B in Figure 5a. One arm (marked BC in the figure) has a fine line etched on it, with the pivot point B lying on the line, and the other arm (AB) is slotted along its length, the extended slot intersecting BC in B. In the slot there rides a peg P; when the position of the peg P is fixed, the

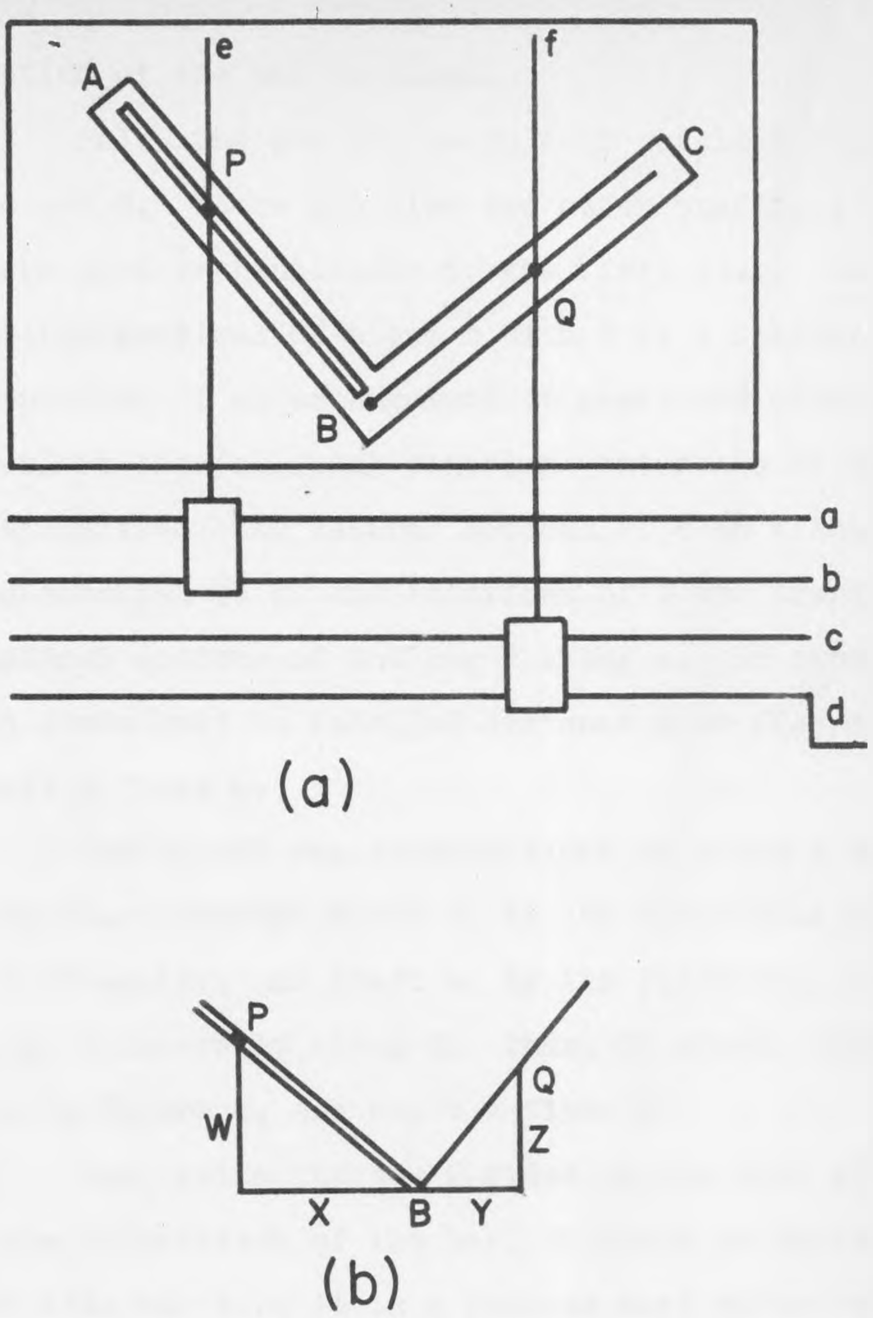

FIG. 5.

orientation of the bar is fixed.

Below the bar are shown four parellel shafts, a, b, c and d. There are also two other shafts, e and f, which are each perpendicular to the first four. The box at the intersections of a and b with e is a schematic representation of an arrangement of gears and other gadgets that perform the following function: rotations of shaft a are translated into lateral motions of e in either direction parallel to a, and rotations of b are translated into lateral motions of the peg P along e. In other words, shaft a fixes what is labelled distance x in Figure 5b, and shaft b fixes w.

The box at the intersections of c and d with f has a similar purpose: shaft d, by its rotational motion, moves f laterally, and shaft c, by its rotations, moves the point Q laterally along f. Thus, in short, shaft d fixes y in Figure 6, and shaft c fixes z.

Now, while the peg P rides in the slot and determines the orientation of the bar, Q is not in physical contact with the bar; it is a pointer that rides near and above the etched line on the bar. In Figure 5a there is shown a crank arm at the end of shaft d; it is by means of this crank that the product is formed and fed into the machine by an operator.

To understand the functioning of the unit, suppose first that some units in the machine are forming the quantities w, x, and z, and that by means of shafts and gears these quantities (all of which are rotations, of course) are made to appear as rotations of shafts b, a, and c, respectively. Shafts a and b will serve to fix the position of the peg P, and shaft c will fix the vertical position of Q along shaft f. An operator, by turning the crank, will drive shaft d in such a way as to keep the pointer Q always directly above the etched line, and thus produce the quantity y of Figure 5b.

By inspection of the similar triangles in Figure 5b, it is obvious that the rotation of shaft d, or the quantity y, is $y = wz/x$.

One notes that in the general case the product of two functions divided simultaneously by a third can be generated. Also, special cases are frequently of interest, and a few are listed below:

(a) By keeping x constant, the product xy alone is produced.

(b) By keeping w or z constant, a simple quotient is produced.

(c) By keeping w and z constant, a reciprocal is produced.

(d) By keeping w constant and connecting y and x by 1:1 gears, the square root of z is produced.

It is evident that ingenuity can discover other uses as need arises.

(d) Function Generators

Function generators, or input tables, are used for feeding information into the machine; essentially they are devices for giving one shaft a rotation having a known functional relationship to that of another shaft. The need for such feed-in of information can arise, of course, when the function is known only as an empirical curve, but is also useful when the production of standard functions by integrators, which is a useful procedure at times, would require the use of more integrating units than are available.

The function is drawn on paper to suitable scale, and the paper is pinned to the input board. As shown in Figure 6, there is a lead screw a running parallel to the board and to the vertical direction on the paper, the screw being supported at its ends by wheeled carriages b and b' that run on rails c and c' parallel to the horizontal direction on the paper.

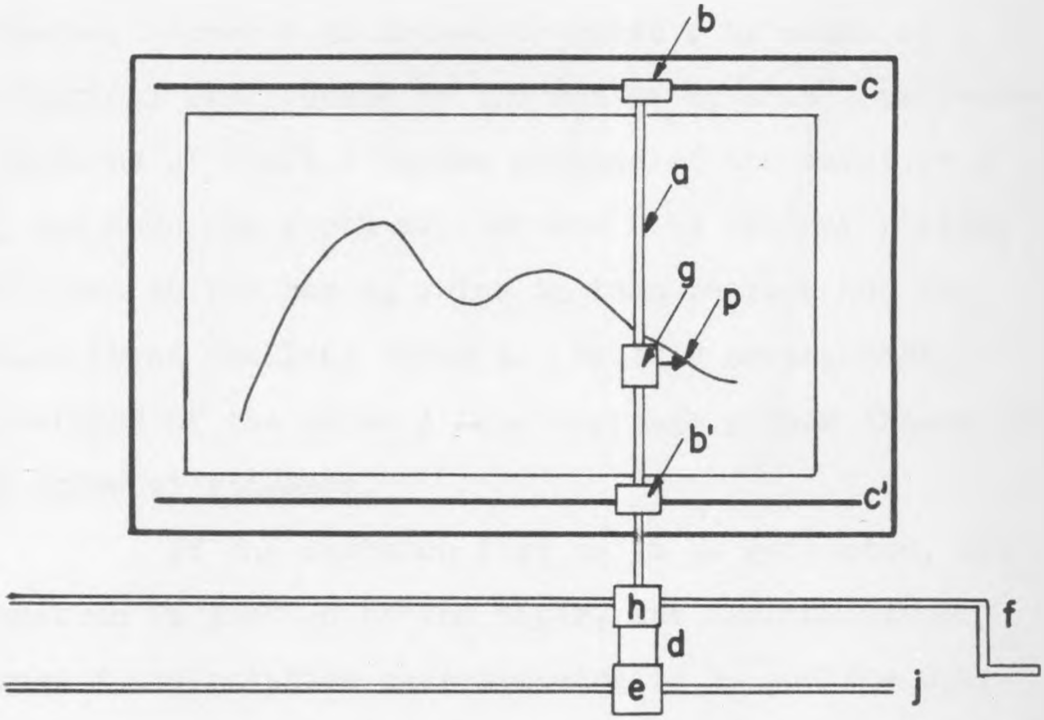

FIG. 6.

A small carriage g carries a pointer p, and has rigidly attached to itself a nut through which the screw passes, so that g rides along the screw as the latter rotates. Screw a is driven by shaft f by means of a mechanical arrangement in the box h; by this arrangement, rotations of shaft f become motions of the carriage g up and down the screw a. The box h is further rigidly attached to the box e, which in turn bears a nut in which turns the lead screw j. By this arrangement, rotations of the screw j move carriage g (and indeed all of screw a) sideways.

If the function f(x) is to be generated, the function is plotted on the paper, the machine gives screw j the rotation corresponding to x, and the operator, by means of the crank attached to shaft f, governs the vertical location of the pointer to keep it strictly on the curve while the horizontal location varies with x. The turning of the crank constitutes a feed-in of f(x) to the appropriate part of the machine; this is accomplished by suitable gears and shafts coupled to shaft f.

(e) Output Table

The output table is exactly the same in design as the input tables, with the sole exceptions that the pointer is replaced by a pencil, and both shafts (corresponding to f and g in Figure 6) are driven by the machine.

What quantity the table is asked to record is a matter of choice; ordinarily the solution x of the differential equation is wanted, plotted as a function of the independent variable. In such a case, the abscissa-shaft may be driven by that part of the machine where the independent variable is available, and the ordinate-shaft by that part where x appears. This latter point will normally be the output of an integrator.

There is a possibility of modifying this design for special purposes. For example, polar coordinate output tables are possible.

(f) Constant Factor Devices

In the list of operating units shown by Miller to be sufficient in any analog computer for the solution of ordinary differential equations, there appears the item Constant Factor Devices. It happens that while this item must appear in the list for conceptual reasons when the

exact nature of the computer is not specified, yet in the mechanical species of the general computer, no special units are needed to play the parts of suppliers of constants before functions.

Constant factors are dealt with in more than one way during the course of set-up design, and this matter will be dealt with at length presently. It is true that spur gear trains function somewhat as constant factor devices, but to explain this feature here would be inaapropriate because its comprehension requires prior explanation of problem set-up design. Accordingly, the question of factors is left open for the present.

(g) Interconnecting System

The main units of the machine have been described; these constitute mechanical realizations of Miller's units. However, as pointed out earlier, means of coupling these units are required. In mechanical terms, what is needed is an interconnecting system of shafts and gears, the suitable arrangement of which will allow a particular differential equation to be presented to the machine.

The general lay-out of such shaft system can best be seen from a diagrammatic plan such as that given in Figure 7. The machine diagrammed there is not any

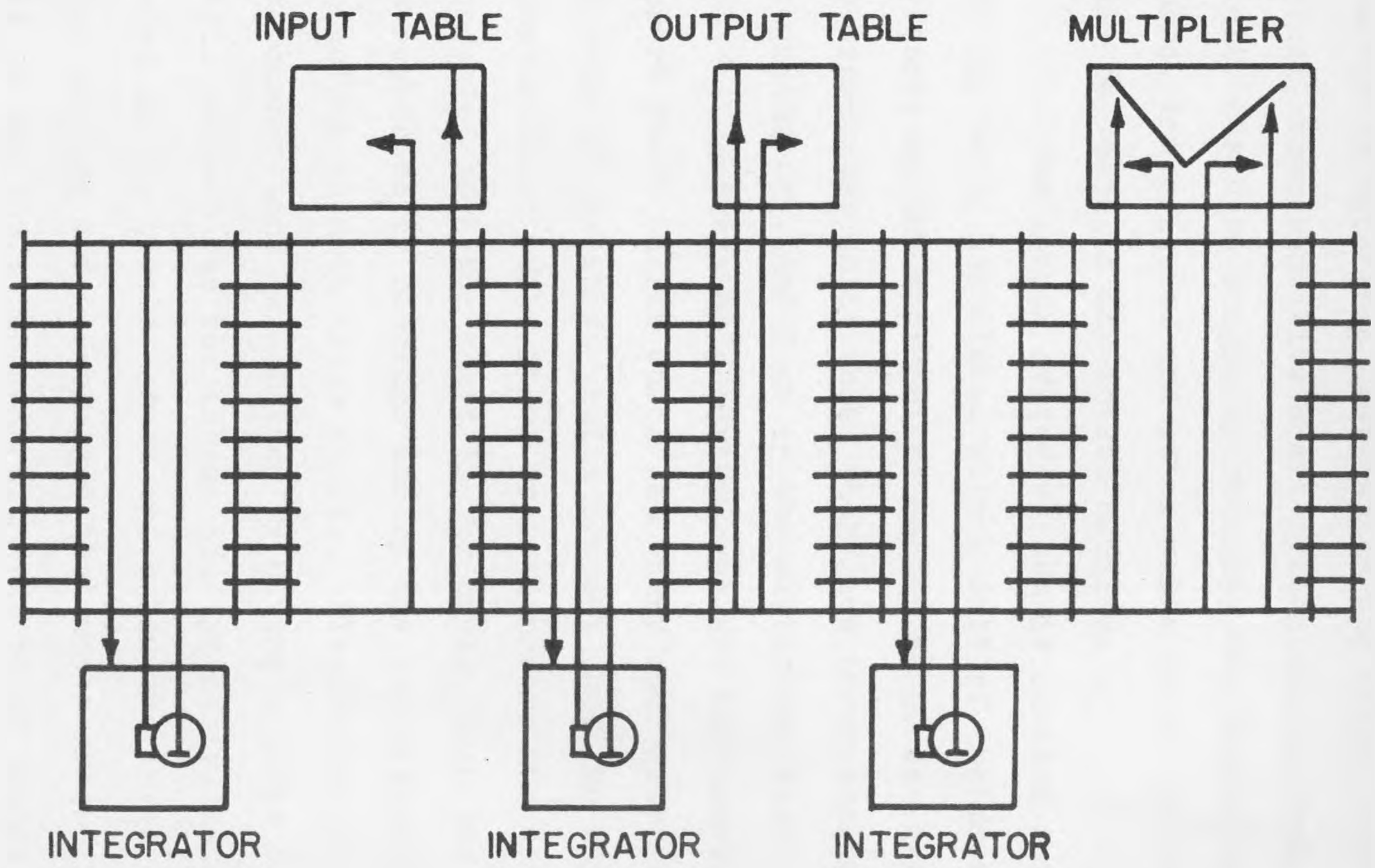
INPUT TABLE
OUTPUT TABLE
MULTIPLIER
INTEGRATOR
INTEGRATOR
INTEGRATOR

FIG. 7.

actually in existence, but rather the figure shows a typical, representative shaft system arrangement. Simplicity has been sought by reducing the number of input tables, integrators, and other units below the number that would be found in any actual machine.

The length of the system is divided into bays, each bay being associated with a unit of the machine. Each bay contains the appropriate number of cross-shafts for the associated unit; thus, there are three shafts in each integrator bay, and four in the multiplier bay.

Between neighboring bays are bus boxes, which support short shafts which can carry spur gears if desired. The ends of the short shafts project into the bays on either side so that either or both protruding ends can be coupled to a bus shaft in the bays. Such bus shafts are supported by bearings and by the helical-gear boxes connecting them to cross shafts. Flexibility in making interconnections is achieved by having a large number of shafts in each bus box (about 35, typically) usually spaced on two levels, above and below the level of the cross shafts, and so arranged that one member of a gear pair on any shaft can mesh with the other member of the pair on any one of four shafts.

Gear wheels are attached to shafts by set screws. Right angle couplings are made by helical gears, and the fact that such gears may be had in right or left handed varieties makes possible an easy choice of positive and negative signs, corresponding to choice of senses of rotations. The significance of this will become clear during the discussion of set-up design.

When all the units needed have been interconnected suitably, some one of the shafts must be rotated by an external motor, in order to cause the machine to run. Invariably, the shaft representing the independent variable will be the shaft so driven. It is desirable to have a variable speed motor serve as the driving force, in order that the most advantageous operating speed for the machine can be used. For example, it often happens that the operator may wish to change to a low speed when a steep portion of some input curve is reached.

(h) Torque Amplifiers

All the descriptions have now been given of those parts of the machine, an understanding of which is prerequisite for a reading of Chapter II. However, discussion of one auxiliary device which is vital to the operation of the machine must yet be given, although this device does

not figure at all in the conceptual scheme of the computer. This device is the torque amplifier.

In the integrators the integrating wheel rests of its own weight on the disc, and a frictional drag typically of the order of one ounce-inch or so exists, and this serves to guarantee that the wheel will follow the rotation of the disc without slipping. It is the case that with a fairly heavy wheel on a reasonably rough disc (say, of frosted glass) negligible slipping, if any at all, occurs, as long as the wheel and its attached shaft are not subject to any drag other than the small amount in the ball bearings supporting it. However, since the rotations of the wheel shaft represent an integral that must be taken in and utilized by some other units of the machine, it is necessary that this shaft be coupled to at least one other unit in the machine. This will entail a drag on the shaft of the order of several pound-inches at least. It is to step up the available torque, of about an ounce-inch, to something at least of the order of 100 pound-inches or so, that the torque amplifiers are introduced. That so large a final value for the torque is sought is due to the fact that one wants to be sure that enough is available so that one need not fear overloading the units. The amplifiers are of such a nature that the

maximum output torque is a latent value, and only so much is actually generated as is needed from moment to moment.

A torque amplifier must not only provide a large output torque, but also is required not to drag on the integrating wheel, and yet to follow faithfully all rotations of that wheel. The mechanical device that fulfills all these requirements is, basically, a form of the common ship's capstain;the development of the unit in the form to be described here is evidently due to C. W. Nieman.[9]

In this device, a small force applied to one end of a friction band wound around a rotating drum produces a greatly increased tension at the other end of the band. For a perfectly flexible band a simple calculation shows the ratio of these tensions to be $e^{f\theta}$ where f is the coefficient of sliding friction and θ is the wrap of the band on the drum. Figure 8 shows a single stage torque amplifier. The input shaft is the shaft attached to the integrator wheel, and the output shaft goes ultimately to a cross shaft in the interconnecting shaft system, through gears which allow for a difference in height of the amplifier and the cross shafts, and give a mechanical advantage of two or four, usually, in addition.

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9. Nieman, C. W. American Machinist, 66 (1927), 895.

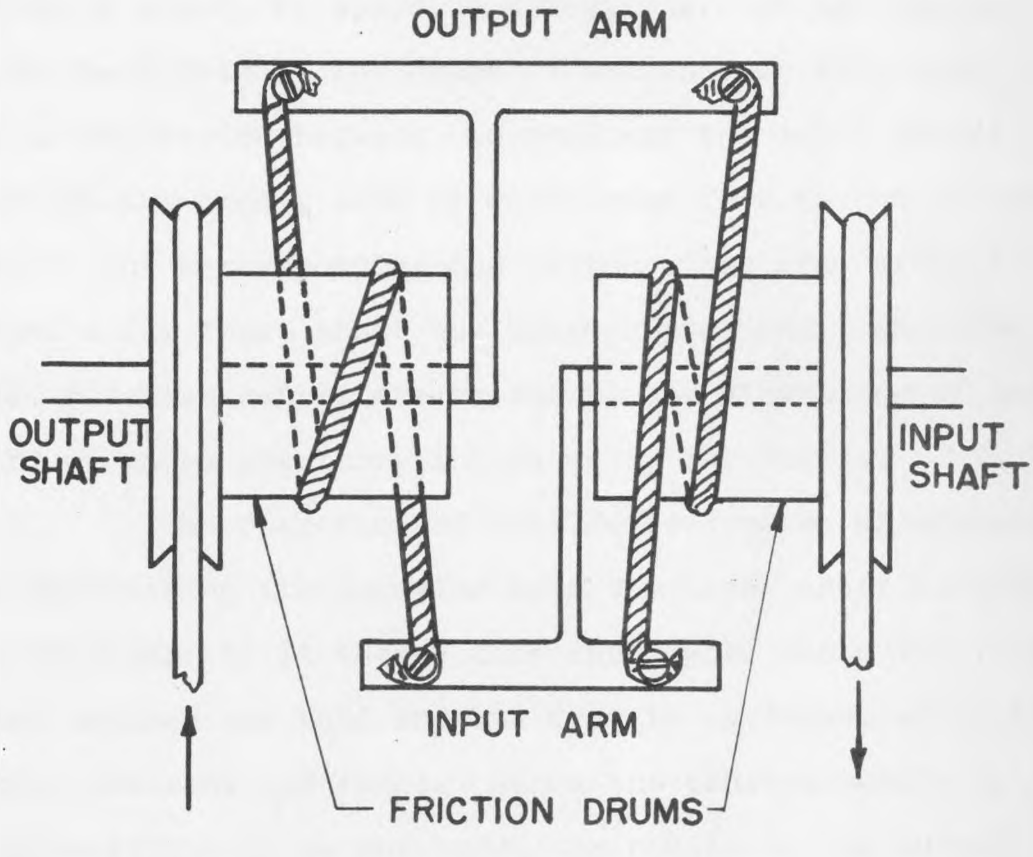
OUTPUT ARM
OUTPUT SHAFT
INPUT SHAFT
INPUT ARM
FRICTION DRUMS

FIG. 8.

The two friction drums rotate about axes coincident with the input and output shafts, but are nowhere in contact with those shafts, and rotate independently of them. These drums are driven at a constant rate by a separate motor, in opposite directions. At the end of each shaft between the drums is attached an arm; there is no connection between the arms and the drums except through the bands, each of which runs from an end of one arm to the corresponding end of the other arm, after being wound a few times about the appropriate drum. When the drum rotation senses are as shown, the directions of the band windings are those indicated in the figure.

The operation of this device can be understood by considering its behavior when the input shaft is given a rotation. As it turns, this shaft will cause the friction between one band and its drum to increase, while the other slackens and idles. Since the tension builds up exponentially along the band, the couple on the output shaft is much greater than the initial torque at the wheel. This operation is independent of the sense of rotation of the input shaft.

The amplification given by a single stage unit might typically be from 50 to 150, but when two stages are connected in series, something of the order of 10,000

can be achieved, and indeed this is the figure commonly sought in the large machines. In the State University of Iowa version, the amplification from two amplifier stages in series is about 5,000.

The power supplied to the output shaft comes from the amplifier motors, of course, and it is possible to think of the unit itself as simply an easily actuated control device.

Chapter II

SET-UP DESIGN

As has been suggested already, before the differential analyzer can be applied to any problem, that problem must be formulated in a suitable way. Preparing a differential equation for the machine will always involve exploratory work on paper, and as might be expected, this work may be considerable in certain more difficult problems. Typically, an equation of a fairly well-behaved nature could be prepared for the machine by an experienced operator in one or two afternoons of work, and the actual labor of putting shafts and gears in place and setting all initial conditions will usually require three hours more. The time needed for preparing suitably scaled input graphs must also be taken into account when the total time needed for putting a problem on the machine is being estimated.

In the present chapter it is proposed to explain the process of designing a machine set-up for a problem. The procedure is a modified form of a generally used one, the modifications having been suggested by experience with the SUI machine as well worth while. Other methods exist, and when the idea has been grasped,

the operator of a computer is free to introduce any innovations he likes.

It is again intended to carry out the explanation by means of an example, for the sake of brevity and clarity. The example, specially suitable for this purpose because its solution can be made to require the use of at least one of each of the units in the differential analyzer without being too involved, will consist of the complete preparation of a plan for solving the equation whose solutions are the Hermite polynomials $H_n(t)$:

$$\ddot{H}_n - 2t\dot{H}_n + 2nH_n = 0 \tag{1}$$

Here n is an integer that fixes the order of the polynomial solution, and in the detailed planning that follows, n will be taken to be 2 for definiteness. However, as will be pointed out presently, the set-up to be designed will be such that a change to different value of n could be easily made, and thus a whole family of Hermite polynomials could be gotten, corresponding to a series of values of the parameter n.

Obviously, there is no need to use a differential analyzer to solve this equation, but this fact in no way diminishes the value of the equation as an example;

the use of an equation with well-known solutions is indeed to be preferred for instructional purposes.

(1) First Schematic Diagram

The first step is always to draw a set-up diagram showing the kinds of units to be used and the general way in which they are to be interconnected. Actually, it may be desirable to make some preliminary transformations of the equation itself, such as simplifying changes of variables, but it is assumed here that such changes have been made if needed.

The first diagram is a purely qualitative one. In it no indications of scale factors, gear ratios, or directions of rotations are included, these things being taken care of in later stages of the planning procedure.

Before proceeding, reference should be made to Figure 9, which shows a set of conventional symbols, originally introduced by Bush[10], which will be used throughout this paper. The figure is self-explanatory, but comment should be made concerning the representations of helical gears and spur gears. Helical gears can be

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10. Bush, op. cit.

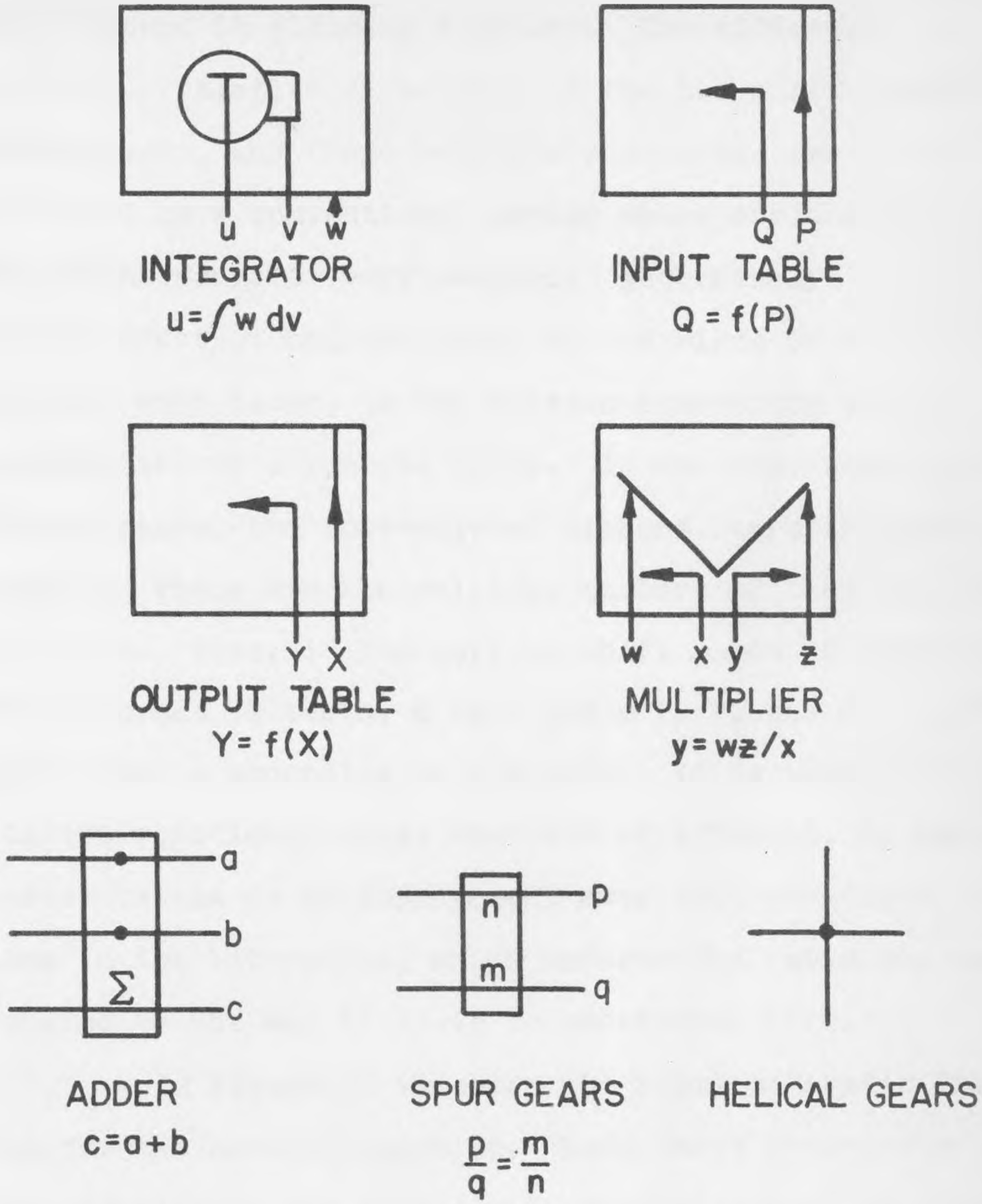
u v w
INTEGRATOR
u=∫w dv
Q P
INPUT TABLE
Q = f(P)
Y X
OUTPUT TABLE
Y= f(X)
w x y z
MULTIPLIER
y= wz/x
a
b
Σ
c
ADDER
c=a+b
n
m
p
q
SPUR GEARS
p/q = m/n
HELICAL GEARS

FIG. 9.

right handed or left handed and this fact must be taken into account in planning a set-up. The difference will affect the relative directions of the two shafts connected by the gears, and these relative directions are to be indicated by a conventional method whose explanation at this point would be very awkward. Accordingly, the matter of the effect of helical gears on the signs of rotations is dealt with later, in the section concerning the introduction of algebraic signs. In the representation of spur gears, the conventional diagram bears the numbers m and n; these are the relative numbers of teeth in the two gears. Thus, if the gear on shaft p has 48 teeth and that on q has 32 teeth, n is 3 and m is 2, and q will turn faster than p according to $q = pn/m$. It is usually the relative rotational rates that are of interest, so this representation is an improvement over that sometimes found in the literature, which inverts the ratio m:n as compared to the way it is to be understood here.

In Figure 10 is shown the first schematic diagram for the Hermite equation. Each shaft represents some quantity in the equation by its rotations, on a scale to be determined later. The bus shaft nearest the units is labelled t; this shaft is driven by the motor, and

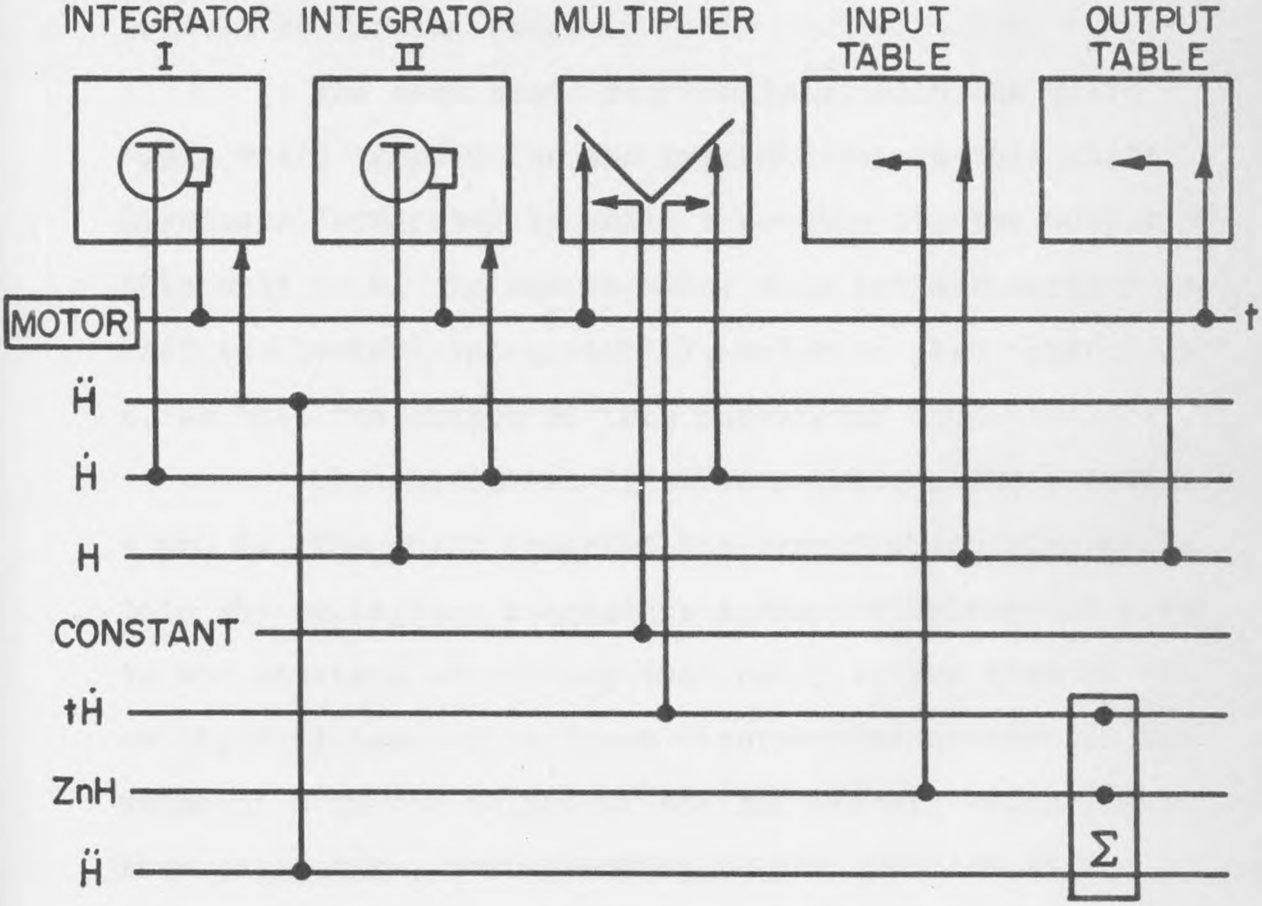
INTEGRATOR I
INTEGRATOR II
MULTIPLIER
INPUT TABLE
OUTPUT TABLE
MOTOR
t
Ḧ
Ḣ
H
CONSTANT
tḢ
ZnH
Ḧ
Σ

FIG. 10.

represents the independent variable. As already mentioned, this shaft can be driven at any convenient speed, or even at varying speeds.

The next shaft represents $\ddot{H}$, with the scale again being ignored for the present. Since this shaft displaces Integrator I, while t rotates it, the output of this unit is $\dot{H}$. As can be seen, this latter quantity is used to displace Integrator II, which is also rotated by t, so that the output of this integrator is H.

The multiplier is shown receiving the quantities t and $\dot{H}$. The shaft labelled "constant" which also feeds into the multiplier represents a fixed displacement given to the abscissa of the peg that rides in the slot of the multiplying bar. This fixed displacement determines the value of x in the output of the multiplier, which is, in this case, $t\dot{H}/x$, corresponding to the notation wz/x used in the section describing the operation of the multiplier. In accordance with the practice of neglecting all scale factors in this first schematic diagram, the cross shaft representing the output of the multiplier is labelled simply $t\dot{H}$.

The input table is used to generate the function 2nH. This function is, to be sure, a very simple function of H, but the use of an input table to generate

it is especially desirable because with this set-up one could get a whole family of Hermite polynomials by having drawn on the input table paper a family of lines with slopes 2n for various values of n, and by running the machine through successive solutions, each time having the input table operator follow a different one of the lines. Since n will be taken to be 2 here, some of the scale factors that will be obtained will spring from this choice, and it should be pointed out that if one were really planning to solve for a whole family of the polynomials, account would have to be taken of the changes in size of some of the quantities entering in the equation when n is changed. Without question, the scale factors and gear ratios used then would not be those used for a solution when n is 2. However, the manner in which the more general problem could be handled will be clear when the procedure for the case n=2 has been understood.

An adder is indicated as taking in $t\dot{H}$ and 2nH and furnishing the sum, which is then connected back to the shaft labelled $\ddot{H}$ by means of a cross shaft and helical gears. If the second and third terms on the left in equation (1) be transposed to the right member, it will be seen that the equality between $\ddot{H}$ and the sum of $t\dot{H}$ and

2nH, which is enforced by this last cross shaft connection, is the same as that required by the equation, aside from signs which are being neglected at present.

Finally, the output table is driven by t and H, so that the graph drawn out during the course of the solution will be directly $H_n(t)$ on a certain pre-arranged scale.

In very complicated problems, especially when the number of available operating units is rather small, as in the SUI machine, some ingenuity must be exercised to produce a good first diagram, for the ease of planning and making the machine set-up and to some small extent the precision finally attained, will depend on this first schematic diagram.

(2) Second Schematic Diagram

In the second stage in the planning of a set-up the scale factors, gear ratios, and constant factors introduced by integrators, adders, and other units, are considered but algebraic signs are still neglected.

Figure 11 shows a completed second stage plan for the present problem. Numerical factors, indicated throughout by letters, are of three kinds: those that

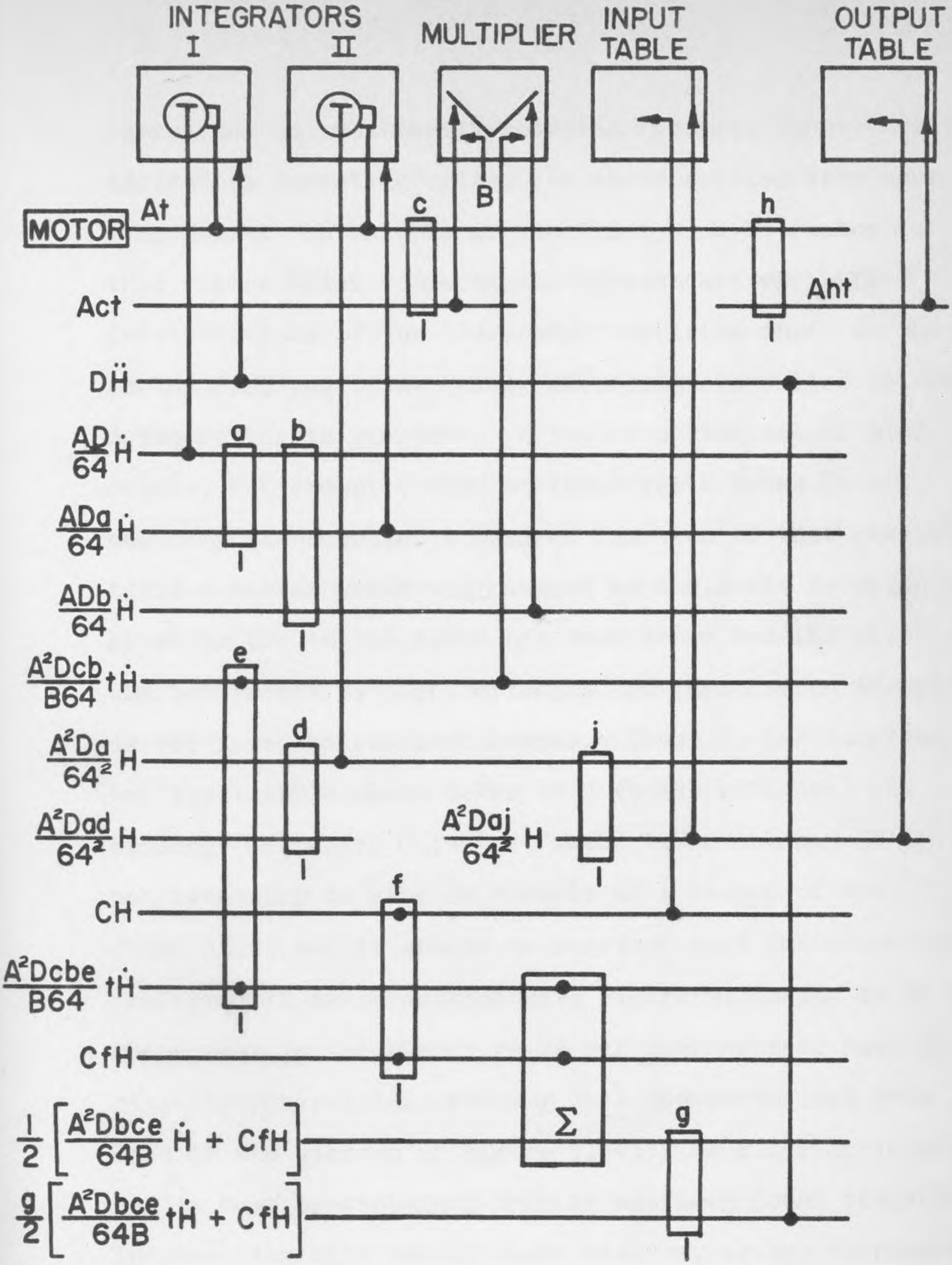
INTEGRATORS
I
II
MULTIPLIER
INPUT TABLE
OUTPUT TABLE
B
MOTOR
At
c
h
Act
Aht
$D\ddot{H}$
a
b
$\frac{AD}{64}\dot{H}$
$\frac{ADa}{64}\dot{H}$
$\frac{ADb}{64}\dot{H}$
e
$\frac{A^2Dcb}{B64}t\dot{H}$
d
j
$\frac{A^2Da}{64^2}H$
$\frac{A^2Dad}{64^2}H$
$\frac{A^2Daj}{64^2}H$
f
CH
$\frac{A^2Dcbe}{B64}t\dot{H}$
CfH
Σ
g
$\frac{1}{2}\left[\frac{A^2Dbce}{64B}\dot{H} + CfH\right]$
$\frac{g}{2}\left[\frac{A^2Dbce}{64B}t\dot{H} + CfH\right]$
1

FIG. 11.

correspond to arbitrarily selected factors, those that are affixed by operating units, and those arising from spur gear pairs. An example of the first kind of factor is that factor which converts the independent variable t into rotations of the independent variable shaft -- thus, in the diagram, these latter rotations are called At, with A the factor in question. A factor of the second kind occurs, for example, when an input table takes in a quantity and supplies a desired function of that quantity times a factor which will depend on the scale to which the graph on the in put table has been drawn and the pitch of the lead screw by means of which the function on the graph is fed into the computer proper. Thus, in the diagram, the input table shown takes in H (times a factor) and produces CH, where C is the factor in question. It is not necessary to give an example of a factor of the third kind, but it should be stressed that the convention referred to in connection with Figure 9, having to do with the manner in which spur gears are represented, must be strictly adhered to. The way this convention has been used in the diagram of Figure 11 will be clear on inspection, but it must be explained that it has been found advisable, in preparing this second stage diagram, always to represent the spur gear ratio as that of a number to unity, rather

than as that of one number to another not unity -- as, a:l rather than m:n -- for this reduces materially the multiplicity of undetermined numbers in the set-up plan.

A further convention not previously mentioned has been found convenient, in the mumerical work that comes later, as an aid in identifying factors of the various kinds. Factors of the first or second kind are always indicated by capital letters, and those of the third kind by small letters. It is not usually necessary to use distinct kinds of symbols for factors of the first and second kinds because factors of the first kind are comparatively rare; in fact, there is usually only one such in any set-up.

The diagram of Figure 11 was prepared by following the first stage plan of Figure 10, beginning at the independent variable shaft, following the first stage plan completely around, everywhere inserting factors of the first and second kinds where appropriate, and, wherever two cross shafts attach to a single bus shaft in Figure 10, always inserting a pair of spur gears so that either cross shaft always connects to the other with a change in scale given by a factor of the third kind. This latter step is carried out to allow for a possibly necessary change of factor; even though later developments

often show many of these gear pairs to be unnecessary, they must be put in when drawing the second stage diagram, for it is very difficult to insert them later if they turn out to be needed and were not included.

The first shaft nearest the integrators is labeled At: this inserts the arbitrary factor converting the mathematical quantity t into the machine quantity At. This quantity, At, is the variable of integration for the two integrators, and fixing the value of A will be equivalent to choosing the total time of running for the machine during the solution. To allow for possibly needed changes of scale before the time shaft rotations can be used in the multiplier and output table, the spur gears with the ratios c:1 and h:1 are inserted as shown.

The input to the first integrator is labeled $D\ddot{H}$. The reason a factor is required here is that the mathematical quantity $\ddot{H}$ may attain so large a value at some time during the course of solution that if it were represented directly by one rotation of the input shaft to one unit of $\ddot{H}$, the integrator disc might be run laterally so far as to cause the integrating wheel to drop off it. The factor D thus serves the purpose of restricting the machine quantity $D\ddot{H}$ to a suitable maximum size.

Once the integration variable and integrand are

decided on for integrator I, the output is fixed. This output has been shown as $AD\dot{H}/64$; the factor A appears via the variable of integration, the factor D via the integrand, and the factor 1/64 has been assumed to be the integrator constant.

This output from integrator I is multiplied by the factors a and b, and is then fed into integrator II and into the multiplier. The reason for inserting the first pair of spur gears is to restrict the maximum displacement of integrator II in the same way as was necessary for integrator I, and the reason for the second pair of spur gears is that a similar limitation of maximum travel must hold for the multiplier, as will be discussed in detail later.

At this point, then, the output of integrator II is fixed. A further factor of A/64 appears as the result of the integration, and $\dot{H}$ has become H. Consequently, the output is $A^2DaH/64^2$.

The inputs to the multiplier have been provided for. In the output the factor 1/B arises as a consequence of the manner in which the value of the constant coordinate of the peg in the multiplying bar affects the output, as explained previously.

The quantity produced by the input table is shown simply as CH, and this fact, while already mentioned, must be commented on with greater detail here. The input to this table is $DA^2adH/64^2$, but it happens, due to the nature of this unit, that the function produced can be regarded as bearing a factor that can be considered independent of the factor attached to the input variable. While the factor attached to H when it goes into the table serves to restrict the machine quantity to a suitable maximum value so that the input table carriage cross-travel will not exceed the width of the graph paper, the factor C arises only from a choice of scale of ordinates on the graph, and the pitch of the operator's lead screw. Reflection on this matter will show that this choice of ordinate scale is independent of the abscissa scale except only that the two scales together ought not to result in a figure too steep to be followed conveniently. The lead screw pitch is certainly fixed and so it too is independent of the abscissa scale. (It ought also to be pointed out that C, as used here, includes the quantity 2n, which is actually 4 in our case.)

The gears e and f serve to make possible an equalizing of the factors attached to the inputs to the adder; that this is necessary has already been explained.

The adder output is indicated as bearing the further factor 1/2, which arises in a way also explained before.

The gear pair g is inserted in order that the factor before the sum coming from the adder may be adjusted to agree with the factor D in $D\ddot{H}$ in accordance with the original equation (1).

Finally, the quantity $DA^2aH/64^2$, which contains the solution sought, is passed through the gear pair j; this serves to reduce or enlarge the scale on which the solution will be recorded. It is generally desirable to have the solution come out on as large a scale as is compatible with the dimensions of the output unit. When the programming procedure has been completed, the scale of both axes in the output graph will be definitely known, and interpretation of the answer can be immediately and directly made without further computation, unless the plotted answer is related to the answer really wanted through some change of variable made before giving the machine the problem.

(3) Evaluation of Numerical Factors

The next step in the set-up design is the determination of all those factors so far indicated only in a general way by letters. The idea involved is that

is that consideration of the machine units, conjointly with consideration of the behavior of the quantities appearing in the equation, give rise to a set of equalities and inequalities, all of which must be satisfied. Satisfaction of the set of relations fixes the values of the quantities involved. Ordinarily the number of equalities and ineqalities is larger than the number of unknown factors, but due to lack of independence among the relations, only most, and not all, the factors are determined by them. The remaining arbitrary values can always then be chosen freely, as long as certain principles to be given presently are satisfied.

The writer has found it best sometimes to convert inequalities into equalities by a judicious decision, before making the evaluation of factors, since this introduces a desirable definiteness that is lacking when inequalities must be handled.

One begins again with the independent variable shaft. Here an equation is supplied us immediately by the fact that one knows what range of the mathematical variable t is to be used. For definiteness, let it be assumed that the solution is to be carried to the point t=4. Then the equation referred to is

$$4A=T \qquad (2)$$

where T is the total number of independent variable shaft rotations made from the start of solution to that point corresponding to the value 4 for t. This equation connects the two numbers A and T, so that when T has been chosen according to criteria to be explained presently, A will be determined.

Next, the fact that the maximum displacements allowable for the integrator carriages is limited supplies two equations. Actually, one gets two inequalities directly, but this is an instance where conversion to equalities is advisable. The inequalities can be expressed as

$$D\left|\ddot{H}\right|_{max} \doteq 120 \quad \text{and} \quad \frac{DA_a}{64}\left|\dot{H}\right|_{max} \doteq 120 \,. \tag{3}$$

Here the subscripts max indicate that the quantities to which they are affixed are the maximum values of those quantities during the period of solution. The vertical bars signify absolute values as is usual.

In these relations, the number 120 is the largest number of rotations of the input shaft to either integrator that can be permitted. This figure refers to the State University of Iowa machine, and was arrived at by multiplying the number of turns of the integrator carriage lead screw required for a travel of one inch, by the

distance from the center of the integrator discs to a point a short distance from the rim. (The former number is 20, and the latter, 6.)

The equalities by which these inequalities can be replaced are

$$D|\ddot{H}|_{max} = 120 \quad \text{and} \quad \frac{DAa}{64}|\dot{H}|_{max} = 120. \qquad (4)$$

Besides the advantage that equalities fix scale factors with definiteness, there is the further important benefit gained that one is now assured that during the solution the whole of the integrator discs will be traversed, instead of only a small region near the center. Restricting traverse to a region near the zero points of the integrators tends to increase the effect of inaccuracies, as will be made clear later in this paper.

The multiplier unit, because of the limited lengths of the slot and line in the multiplying bar, requires that a number of inequalites be satisfied. In the State University of Iowa machine, the length of travel from one end of the slot to the other end, or from one end to the other of the line, corresponds to 184 turns of the input shafts or output shaft. Thus, if the machine quantities going in and coming out of the multiplier are not to carry the peg or pointer too far, the maxima of all those quantities must be not more than 184, in terms of rotations.

In addition, the fixed quantity B, which is a number of rotations, must not be greater than 184. Furthermore, if the slot (or line) be regarded as the hypotenuse of a right triangle (as in the two triangles with sides x and w, and y and z, in figure 5), then the sum of the squares of the sides in either triangle expressed as rotations, must not exceed 184^2.

In this way one gets:

$$
\left.
\begin{array}{ll}
Act_{max} \leq 184 & \dfrac{DAb}{64} |\dot{H}|_{max} \leq 184 \\[2ex]
B \leq 184 & \\[2ex]
B^2 + (Act_{max})^2 \leq 184^2 & \dfrac{DA^2cb}{B64} |t\dot{H}|_{max} \leq 184 \\[2ex]
\left[\left(\dfrac{DAb}{64}\dot{H}\right)^2 + \left(\dfrac{DA^2cb}{64B}t\dot{H}\right)^2\right]_{max} \leq 184^2 &
\end{array}
\right\} \quad (5)
$$

It is not necessary to inquire into the degree of independence of these relations. It is enough to know that they always are consistent for suitable choices of the quantities in them.

Next, one turns one's attention to the input table. Two inequalities arise here from the requirements that the range of ordinates and abscissae be confined to

the graph paper. There is also a need for avoiding too steep slopes in the curve, but it is best not to incorporate this into the system of inequalities. In practice, when excessively steep curves occur they often are unavoidable (as in the case of functions resembling step functions), and in any case the variability of the motor speed in available as a means of coping with steepness. For the State University of Iowa machine the numbers of shaft rotations from the center of the axes of abscissae and ordinates to the ends of those axes in either direction are 260 and 200 respectively. Thus one gets

$$\frac{DA^2ad}{64^2}\,|H|_{max} \leq 260 \qquad \text{and} \qquad C\,|\dot{H}|_{max} \leq 200\,. \qquad (6)$$

The adder, by its nature, requires an equality of the factors before the addends. In the present case one must have

$$\frac{2t\dot{H}}{4H} = \frac{DA^2ceb}{B64}\,\frac{t\dot{H}}{cfH}\,, \text{ or } \qquad Cf = \frac{DA^2ceb}{32B}\,. \qquad (7)$$

When the adder is coupled to the input of the first integrator, the ratios between the machine quantities must be the same as those between the corresponding mathematical quantities in equation (1). That is,

$$\ddot{H} : 2t\dot{H} : 4H = D\ddot{H} : \frac{g}{2}\,\frac{A^2ceb}{64B}\,t\dot{H} : \frac{g}{2}\,CfH\,. \qquad (8)$$

The relations (2) through (8) comprise the entire set that must be satisfied, except that two more exist, involving the numbers h and j. The reason these latter are considered separately is that these quantities are not concerned in the solution of the problem, but only in recording the answer. After all other factors have been established, h and j can be determined by choosing them to give an output graph of suitable proportions. The maximum travel along the independent variable axis is 480 turns, and for the ordinates (from center to either extreme), 200 turns, in the State University of Iowa machine. Thus one has

$$hAt_{max} \leq 480 \quad \text{and} \quad j\frac{A^2a}{64^2}|H|_{max} \leq 200 .$$

In this case again, since as large a figure as possible is desirable, it is better to write these relations as

$$hAt_{max} \simeq 480 \quad \text{and} \quad j\frac{A^2a}{64^2}|H|_{max} \simeq 200 . \qquad (9)$$

For convenience in reference, the relations to be dealt with are collected together below:

$$
\left.\begin{array}{llll}
\text{(a)} & 4A = T & \text{(b)} & D\left|\ddot{H}\right|_{max} = 120 \\[2ex]
\text{(c)} & \dfrac{DAa}{64}\left|\dot{H}\right|_{max} = 120 & \text{(d)} & Act_{max} \leqq 184 \\[2ex]
\text{(e)} & \dfrac{DAb}{64}\left|\dot{H}\right|_{max} \leqq 184 & \text{(f)} & B \leqq 184 \\[2ex]
\text{(g)} & \dfrac{DA^2cb}{B64}\left|t\dot{H}\right|_{max} \leqq 184 & \text{(h)} & B^2 + \left(Act_{max}\right)^2 \leqq 184^2 \\[2ex]
\text{(i)} & \left[\left(\dfrac{DAb}{64}\dot{H}\right)^2 + \left(\dfrac{DA^2cb}{B64}t\dot{H}\right)^2\right]_{max} \leqq 184^2 & & \\[2ex]
\text{(j)} & \dfrac{DA^2ad}{64^2}\left|H\right|_{max} \leqq 260 & \text{(k)} & C\left|H\right|_{max} \leqq 200 \\[2ex]
\text{(l)} & DA^2ceb/64B = 1/2\ Cf & \text{(m)} & D : \dfrac{g}{2}\dfrac{A^2cebD}{B64} : \dfrac{g}{2}Cf = 1:2:4
\end{array}\right\} \quad (10)
$$

There are eleven quantities to be determined: A, B, C, D, a, b, c, d, e, f, and g. (T is not counted since A and T are really equivalent.) The number of relations available is evidently greater than eleven, but will not determine all eleven numbers, as already pointed out.

In the use of the relations (10), one is further guided by certain general principles that could in principle be expressed by more inequalities. However, so to write them would be to expand the list of relations greatly

without much helping matters. In practice, one can better merely bear in mind the content of the principles, which are:

(1) Integrators are never to be geared up, if possible; this is to assure small loading on them. This condition restricts the values of some spur gear ratios, such as a, b, and d in our example.

(2) Spur gear ratios resulting in mechanical disadvantages greater than 9 or 10 for the outputs of such units and the multiplier or input tables must be avoided usually. For example, the operator of an input table must not be asked to drive any very heavy load through a stepped-up ratio that is too large. Experience alone will enable one to judge when loads are sufficiently moderate to be usable.

(3) Extreme slow-downs of rotations are to be avoided, usually. If one shaft is going at a reasonable rate, it should not be geared to another through such a ratio that the second shaft goes very slowly. Rates of the order of one turn in 10 or more seconds can be taken as excessively slow, on the whole. The reason for this requirement is that in general errors in the machine are of fixed, absolute size, as is evident in the case of backlash errors, for example, and while these errors are

relatively very small when the number of rotations between changes of signs of rotations is large enough, they can be relatively very large when the latter number of rotations is small.

(4) All spur gear ratios decided on must be available from the stock of gears on hand. A list of available gears for the State University of Iowa machine is given elsewhere in this paper, but for the present it will suffice to point out that the pairs usually give ratios of simple small numbers, as 1:2, 1:3, 2:3.

(5) The total solution time must be neither too long nor too short. It is principally for convenience that one wants to avoid long running times. Short times are to be avoided because, in such cases, either the running speeds of some parts of the machine would have to be excessively high, in order to make the solution so quickly, or else the total number of rotations involved in the various parts of the set-up would have to be small, and this would result in large relative errors.

One more preliminary matter must be discussed here. In the relations (10) there appear the quantities t_{max}, $|H|_{max}$, and other similar ones. These must be known before any scale factors can be found. In preparing a set-up for an actual problem, one will almost always have some

knowledge of the range of variation of the solution and its derivatives, and even when this knowledge is incomplete, one can nearly always make very adequate estimates. A case would be most unusual if one were completely unable to guess these quantities well enough, but even in such an unlikely event, one would not be helpless, for then trial set-ups could be used to find a suitable one, though perhaps in extreme instances this might involve considerable trial and error work.

In the example being considered, the solution $H_2(t)$ is well known, and one has directly from it, on the assumption $t_{max} = 4$,

$$|H|_{max} = 62, \quad |\dot{H}|_{max} = 32, \quad |\ddot{H}|_{max} = 8, \quad |t\dot{H}|_{max} = 128. \tag{11}$$

Now all the information needed for the determination of scale factors and gear ratios is at hand, and we start with equation 10a. The best operating conditions for integrators obtain when the motor is producing a time-shaft rate of about one turn per second, and the cross shafts that rotate the integrator discs are coupled directly to the time-shaft. This knowledge suggest the choice of the value 256 for T. This means about four minutes of running time, and results in a value for A which is a power of two -- this is always an advantage,

because of the ubiquitousness of powers of two in all the calculations that follow. In fact A turns out to be 64, which will cancel the integrator number 64, and thus contribute a simplifying influence in the proceedings.

The experience that suggested four minutes as a suitable period in this case, also indicates, in general, that any simple monotonic solution can be run off in about this length of time, and highly oscillatory solutions should be allotted at least 15 or 20 minutes if more than just one or two cycles are to be produced. In any case, the value of A gotten by picking T must be regarded as tentative; the beginner is apt to find that his first value of A leads to impossible conditions on other quantities.

Relation 10b now fixes D uniquely, then 10c fixes a, and 10d and 10e fix upper limits for b and c. Thus $D = 15$, $a = 1/4$, $c \leq 23/32$, and $b \leq 23/60$. Notice that $a < 1$ means the output of integrator I is geared down, as is desirable.

In 10f, 10g, 10h, and 10i, the quantity B will be the only unknown, once b and c have been chosen. By reworking these relations, one gets successively
$B \leq 184$, $B \geq 640\ bc$, $B \leq \sqrt{184^2 - 256^2 c^2}$,

$$B \geq \frac{2DA^2bc}{\sqrt{184^2 - \left(\frac{DAb}{2}\right)^2}} \quad .$$

In order that the minimum B be considerably less than 184, the product bc should be, say, 1/20 or so, and one might reasonably try b=1/4, c=1/5. With these values, one finds B is required to satisfy $40 < B \leq 170$. The value 64 would seem reasonable, and again has the advantage of cancelling the frequently occuring 64's in our relations.

Equation 10j gives $d \leq 104/93$ and 10k gives $C \leq 200/62$; the choices d=1 and c=3 seem worthy of trial.

10l and 10m, when worked down with all values thus far assumed, produce the three relations e=2f, gf=40, and eg=80. Of these, one is derivable from the others, so one need only deal with, say, the first two. Since gf is to be rather large, and neither g nor f alone can be very large, while it is good to have each an integer to simplify producing their values by spur gear trains, the choices f=4, g=10, and e=8 are well-nigh dictated.

At this stage, the following figures have been decided upon:

A=64	C=3	a=1/4	c=1/5	e=8	g=10
B=64	D=15	b=1/4	d=1	f=4	

That these values are consistent with (10) is certain, but one must also be sure that the general principles given earlier are complied with. A has been chosen deliberately to make the time of solution reasonable and moderate, all the spur gear ratios arrived at are readily available sizes, the outputs of the integrators are not geared up, and no extreme slow-ups of rotational speeds occur. The only remaining question is whether the gears e, f, and g together might not constitute an excessive step-up combination. These gears function in the following way: The multiplier operator must drive one side of the adder through an 8:1 step-up ratio, the operator of the input unit must drive the other side of the adder through a 4:1 step-up, and the adder output is further stepped up by 10 to 1. Of the two operators, one has a 40:1 ratio to deal with, and the other a 20:1 ratio.

These figures imply large mechanical disadvantages for the operators, but it should be noticed that the two operators, between them, share the very light task of displacing integrator I, and do nothing else except overcome bearing friction. This is so small a load that the step-up ratios occasioned by e, f, and g might be considered acceptable.

It must not be supposed that such large ratios as these are ever to be accepted without adequate investigation to assure that some part or some person is not being assigned an impossible task. The example, as worked out here, could have been made to result in different e, f, and g values, but it was felt instructive to have them turn out as they did. The reader would do well to carry out the factor determination for the example treated here in an attempt to improve on this set up, if he wants to see what is involved in more than a superficial way.

To complete the task at hand, h and j must be determined., One gets directly $h \simeq 1.8$ and $j \simeq .9$, and these can be rounded off to h=2 and j=1. This results in a well proportioned graph of dimensions 512 turns by 232 turns, which means about 26 x 12 inches, assuming lead screws of 1/20 pitch.

(4) The Penultimate Diagram

Two matters remain to be taken care of: the problem of adjusting algebraic signs in the set-up, and the problem of deciding precisely how the arrangement of gears and shafts is to be set on the machine itself. It is best to deal with the latter problem first.

A glance at Figure 11 will show that that diagram cannot represent an actual map of the shaft system, for in some places shafts coupled by spur gears are separated by considerable distances, and the overall width of the whole system is excessive. The task, then, is to rearrange the diagram of Figure 11 so that these two defects are corrected. (Also, the adder must be so placed in arrangement that the input shafts are properly located with respect to it.)

Little explanation of how this is done is required, and Figure 12 is given without comment as one configuration that might be arrived at. (The arrows and the letters R and L that appear there will be explained in the next section.) It will be noticed that only mathematical quantities have been labeled; the omission of scale factors is permissible since it now known that these will come out as wanted, and there is no point in submitting to the inconvenience of inserting them everywhere.

(5) Introduction of Algebraic Signs

Unless a system of conventions is adopted and strictly adhered to when dealing with the signs in a set-up, confusion very easily results, and in fact it becomes well-nigh impossible to handle the problem at all.

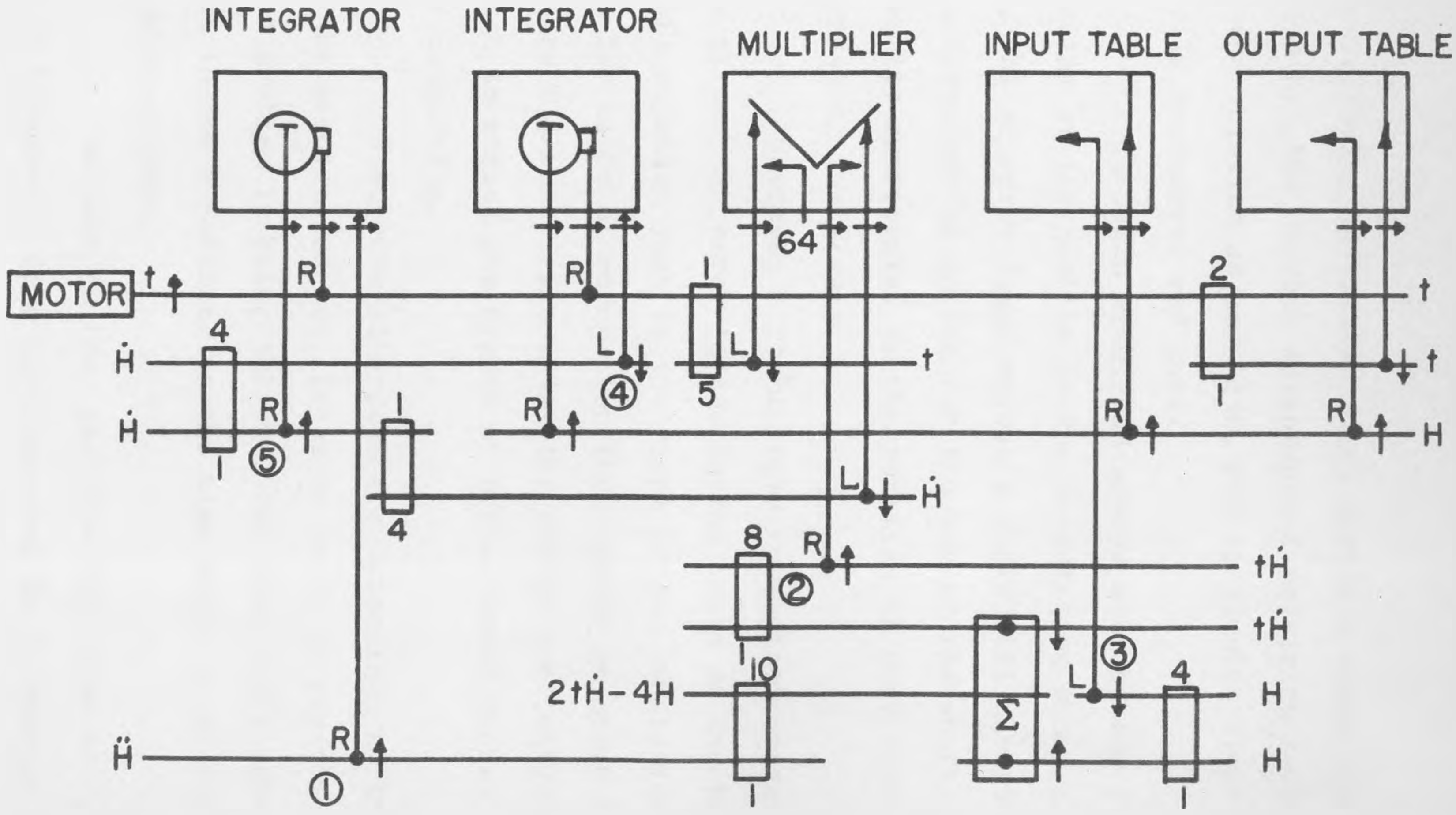
INTEGRATOR
INTEGRATOR
MULTIPLIER
INPUT TABLE
OUTPUT TABLE
MOTOR
64
Σ

FIG. 12.

The following discussion makes use of a consistent scheme which has the supreme advantage of requiring no knowledge of the solution at any time, and is simple, clear, and easily remembered and used.

To begin with, one always assumes one is looking at the machine in the orientation shown in Figure 12, and an arrow drawn across a shaft will always indicate the direction of motion for the top of the shaft when the quantity represented by the rotation of that shaft is increasing positively.

Wherever a right angle connection appears, one places the arrow indicatingthe sense of the horizontal shaft rotation just to the right of the connection point. The two kinds of connection then appear as shown in Figure 13, which also shows the use of the letters R and L to stress the right- or left- handed character of the connection.

Now, several arrows are determined by the nature of the machine and can always be drawn in regardless of what problem is being solved. The time shaft motor always goes in one direction, so the time shaft sense may be labeled at once.

We take as the positive direction of rotation of an integrator disc that imparted to it when it is connected

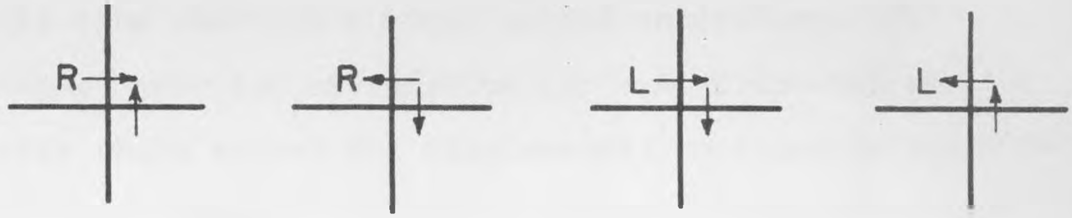

FIG. 13.

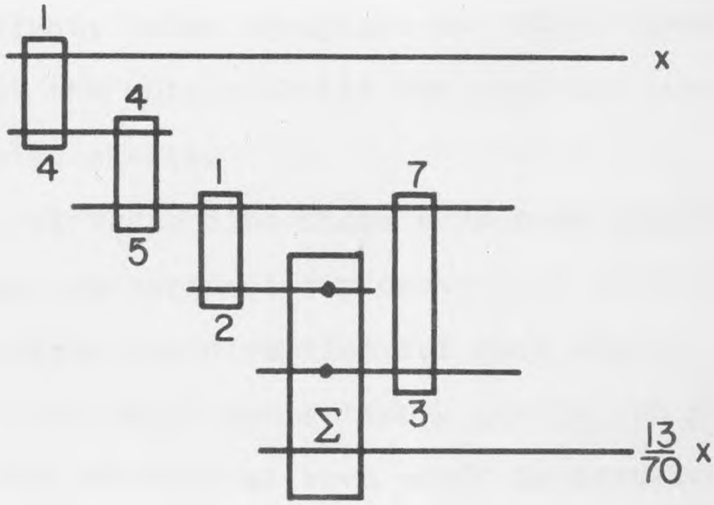

FIG. 14.

to the time shaft by a right handed connection. The positive direction of rotation for a displacement shaft is that which causes the displacement to increase positively. If an integrator is set with a positive displacement and rotation in a positive sense, the positive sense of the output shaft is fixed.

Finally, in connection with the multiplier and input tables, one sets the positive directions by convention. For example, one usually likes the graph on the input table to have its ordinate axis positive direction upward, and the abscissa axis positive direction to the right; then the gears and other elements in the makeup of the unit will fix the positive senses for the associated shafts.

When all these directions have been inserted in the diagram, one can begin at any convenient point and work around, marking the direction for each shaft, and labeling each right angle connection R or L. It should be noted that the rotation of each shaft is reversed on passing through a pair of spur gears, and that if a sum is to produced by an adder, both inputs must have the same sense of rotation.

Some thought given to Figure 12 will be sufficient

to impart an understanding of the procedure, but two points might give some trouble; these will be clarified here. The first occurs in connection with the adder; here the question is how the connections at the points marked (1), (2), and (3) are determined, and how the sign of the adder output can be found properly. In particular, the L connection at (3) needs explanation. The equation says that $\ddot{H}=2t\dot{H} - 4H$, and if one refers to the cross shafts at (1), (2), and (3), one sees that a positive rotation at (2) (corresponding to $2t\dot{H}$) contributes a positive rotation at (1), and a positive rotation at (3) (corresponding to $4H$) contributes a negative rotation at (1). This is in agreement with the signs in the equation. In general, the question of the sign of the adder output can be answered by saying that the positive direction for it can be taken to be in either sense. One arrives at this conclusion by reasoning as follows; Suppose only one selected cross shaft somewhere feeds into one side of the adder, the other addend being held constant. Then the only thing that affects the machine is the relation between the directions of rotations of that one cross shaft and the cross shaft connected to the bus shaft representing the adder output. Thus when the cross shaft at (2) undergoes

a positive rotation, while that at (3) is motionless, the cross shaft at (1) must undergo a positive rotation to agree with $\ddot{H} = 2t\dot{H}$, but it is of no significance whatever whether the adder output be called positive or negative.

The other point that is likely to annoy one unacquainted with these machines occurs at the junction labeled (4). The bus shaft there is labeled $\dot{H}$, but if one refers to the junction (5), one finds the bus shaft there labeled $\dot{H}$ also. Since these two shafts rotate oppositely, should not one be called $-\dot{H}$ rather than $\dot{H}$? The answer is no, because the machine cannot recognize such a labeling -- all it knows is the sense of rotation given the displacement screw of integrator II when the output of integrator one is positive. If one looks at that screw and that output shaft, one sees that they are coupled right-handedly, as required.

When one is labeling a right angle connection, one must look always at the cross-shafts affected by the connection, and not at the bus shafts involved.

(6) Initial Conditions

Planning the machine arrangement has now been completed, and the next step would be to insert bus

shafts and gears into the interconnecting system of the machine, and install the graph on the input table. When this has been done, the machine must be adjusted so that when started, it will begin with the proper initial conditions. This is easily done, and little explanation will be required.

Setting the multiplier is a matter of giving each of the four coordinates in it its proper initial value. Thus, if the solution is to begin with $t=0$, $H=-2, \dot{H}=0$, $\ddot{H}=8$, the quantities called w, x, y, and z in figure 6 are set at 0, 64, 0, and 0, respectively. The input table is set so that the pointer has coordinates corresponding to the values of H and 2nH at $H= -2$. The output table is set at the point chosen to represent $t=0$ and $H= -2$. Each integrator must also be set, and this is done by means of a pointer attached to the carriage, which moves over a scale, and a handle at the end of the displacement screw. Integrators I and II must be given initial displacements of 8 and 0, respectively, in our example.

It will be noticed that changing from one set of initial conditions to another is easily accomplished, and this is advantageous when a family of particular solutions is wanted, and also in another situation that

often occurs. This situation arises when all the initial conditions are given, not at the initial point, but some at the initial point and some elsewhere along the path of integration. In such an event, the question of how the machine is to be started off must be answered. What is done is to use the given initial conditions, and to guess at the rest, and to see how nearly the resulting solution agrees with the remaining given conditions. The ease with which initial conditions can be changed means that this searching procedure can be carried out easily. (Problems in which all the conditions are at one end of the path are sometimes called by the expressive name "marching problems", while those in which all conditions are not at one end are called "jury problems".)

When imposed conditions are of still different types, such, for example, as the normalization condition so common in wave mechanical problems, methods must be resorted to that are considered as beyond the scope of this paper.

(7) Development of Numerical Constants

Before this discussion of the problem of set-up design is complete, there must be added some remarks on the handling of constant coefficients. The number of

such coefficients can always be reduced to a small number by transforming the equation, but some still may remain. It is also clear that sometimes even these can be supplied conveniently by some operating unit -- for example, an input table can be used very neatly to put any factor before any quantity. Also, the manipulation of the scale factors in the set-up design can often, if not always, be used advantageously to simplify or even to eliminate any serious numerical constant problem that might otherwise exist.

But still cases may arise where it is desired to make a shaft rotation be nx, n being constant, when another shaft is available whose rotation is simply x. By a choice of scale factors, n can be made less than one, and it is sufficient to deal with such cases. Also, if n is a number given directly by some pair of gears, no problem exists; thus only values other than 1/2, 1/3, 2/3, and the like, need be considered.

The most direct way to achieve the value of n is by a chain of spur gears. By use of combinations of readily available gear ratios, it is possible to build up a wide range of values of n. A good method for selecting the gears to be used is to set n on a slide rule, and to look for integers whose ratio is near this value and

which can be expressed as a product of gear ratios available.

Sometimes when n is not obtainable by a gear train, it may be the sum or difference of two available fractions. As an example, $n=13/70$ could be built up by the arrangement of figure 14, using an adder.

Often a very good approximation can be gotten by taking out a factor that can be represented by a gear train, leaving a second factor of the form $1 \pm N$, where N is small. Then N is obtained as closely as possible by gears, and $1 \pm N$ is formed with an adder. This method restricts the error to the production of N, and the net effect is that of a smaller error in n. In the example following, the error is in taking 1/4 times 1/4 to be .0626 instead of .0625:

$$.4554 = 3/7(1.0626) \doteq 3/7(1+1/4 \times 1/4).$$

The error of .6% in N gives an error of .008% in n.

Chapter III

CONSTRUCTIONAL DETAILS OF THE S.U.I. COMPUTER

Only a terse description of the actual details of the State University of Iowa computer will be given in this paper; those readers who have access to the machine will gain more acquaintance with its details in a few minutes of inspection than a great many pages would impart, and those who do not have such access will scarcely be concerned with the finer points involved. If any propose to build another such machine, profit would still be derived only from an account of the main features incorporated in the machine.

Accordingly, it is proposed to give a description which will become highly detailed only when details are required to show how some difficulty of genuine interest was resolved. In particular, dimensions are usually not given except when they are of use in problem set-up design. Furthermore, the description will be made yet simpler by making it almost altogether a commentary on the accompanying photographs; a project of the complexity of this one could scarcely otherwise be intelligibly described except at great length.

It is well to begin by outlining the general level on which the planning and construction was carried

out. Originally it was intended to construct the computer at a very minimum cost in money and man-hours of labor, and some experimentation was first done in an effort to find out what such a minimum might be. Parts of some units were put together out of standard Erector Set parts, with the notion that it might be possible to work entirely in this medium with good results. This was found to be quite impossible because of the extreme lightness of the girders and other parts and the unsatisfactory quality of the gears. Then the same parts were remade using common war-surplus gears, 1/4 inch steel shafting, and plate aluminum. It was found that sufficient ruggedness still did not exist, mainly because shafts of this size are still very flexible under the considerable stresses that arise in a complicated mesh of gears and shafts unless the bearings and alignments are extremely carefully made. On the basis of some further investigation, it was decided that, all in all, sufficient economy would be achieved, little enough find machining would be required, and that the solidity of the whole machine would be high enough, if most shafting was of 3/8 inch steel, most gears were standard 48 pitch American Stock steel gears, and most supporting plates were of 1/4 inch aluminum. It also seemed likely that the total frictional drag would not

become excessive if simple bored bearings were used nearly everywhere to bear the shafts; this would contribute to the economy of the project, for ball bearings are fairly costly in quantity.

The foregoing considerations account for the present aspect of the computer; no machining was ever done where it did not materially contribute to performance, and if a part could be made either crudely or comparatively finely with little change in behavior characteristics, the choice of crudeness was invariably made. No effort was made to improve mere appearances for appearances' sake, although inspection of the pictures will show that surprisingly good-looking results can easily be achieved with no extra expenditure of labor.

Total cost of the machine, exclusive of labor, was approximately $600, some two thirds of which represents the cost of gears. Actually, smaller and cheaper brass gears, thinner shafting, common die-cue brass lead screws, and other inexpensive parts were frequently used instead of the heavier and more costly large steel gears and other parts. No one single item other than the gear stock was expensive; the only items that would have been so were the two motors required, but these were available without purchase.

Construction required about thirty hours of work weekly for each of two persons for a period of about eight months, after which one person was occupied with the task for about 20 hours a week for an additional six months. However, many false starts were made, and often completed work was discarded and redone to incorporate better ideas, to reduce drag, and for various other reasons. With adequate planning and guidance of an experienced person, two skilled machinists could probably construct a similar machine in two or three months of full-time work.

Now reference is made to Figure 15, where one sees the computer from a corner adjoining the first integrator. The entire machine is roughly eight by ten feet square, nearly filling a small room, the main horizontal level is 36 inches from the floor, and other dimensions that might be of interest can be estimated by comparison with the foot-long scale on the block adjoining the first integrator at the lower center of the picture.

The reader will readily distinguish the various units of the machine. These include two integrator units, a two-stage amplifier associated with each of these, the output table (behind the integrators), the multiplier (the vertical unit behind the output table), and the shaft

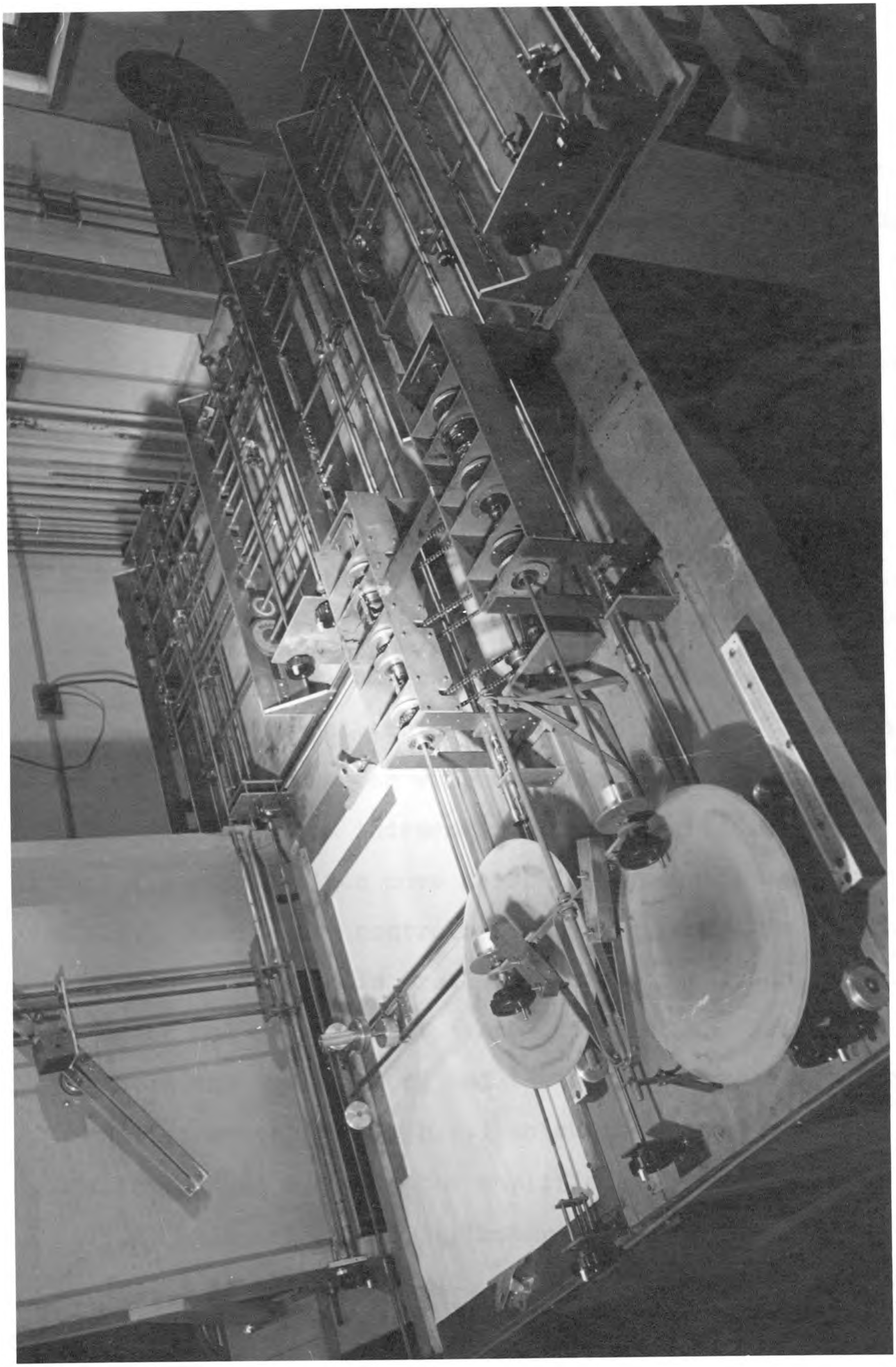

FIG. 15

system. The input tables are located to the right of the shaft system but only one of them is visible in the figure. This unit, which is the vertical apparatus at the upper right of the figure, bears on the reverse side of its vertical plane another input unit just like the one visible, and a second such pair of input tables, back to back, lies just off the right hand central part of the photograph, to give a total of four inputs.

Figure 16 gives a view of the machine from a vantage point on the opposite side of the machine. Now the operator's side of the multiplier can be seen, as can one of the input tables lacking in Figure 15. At the end of the shaft system will be noticed the 1/2 h.p. motor which serves as the independent variable drive. In order that a supervisor can move about freely during operation without leaving the controls far behind, a double switch box on a long cable is provided; this can be seen hanging from the end of the shaft system. One of the switches controls the drive motor, and the other controls the torque amplifier motor (of 1/2 h.p.) which is mounted beneath the table that supports the amplifiers.

In order to illustrate some of the discussion to follow in greater detail, Figures 17 and 18 are included.

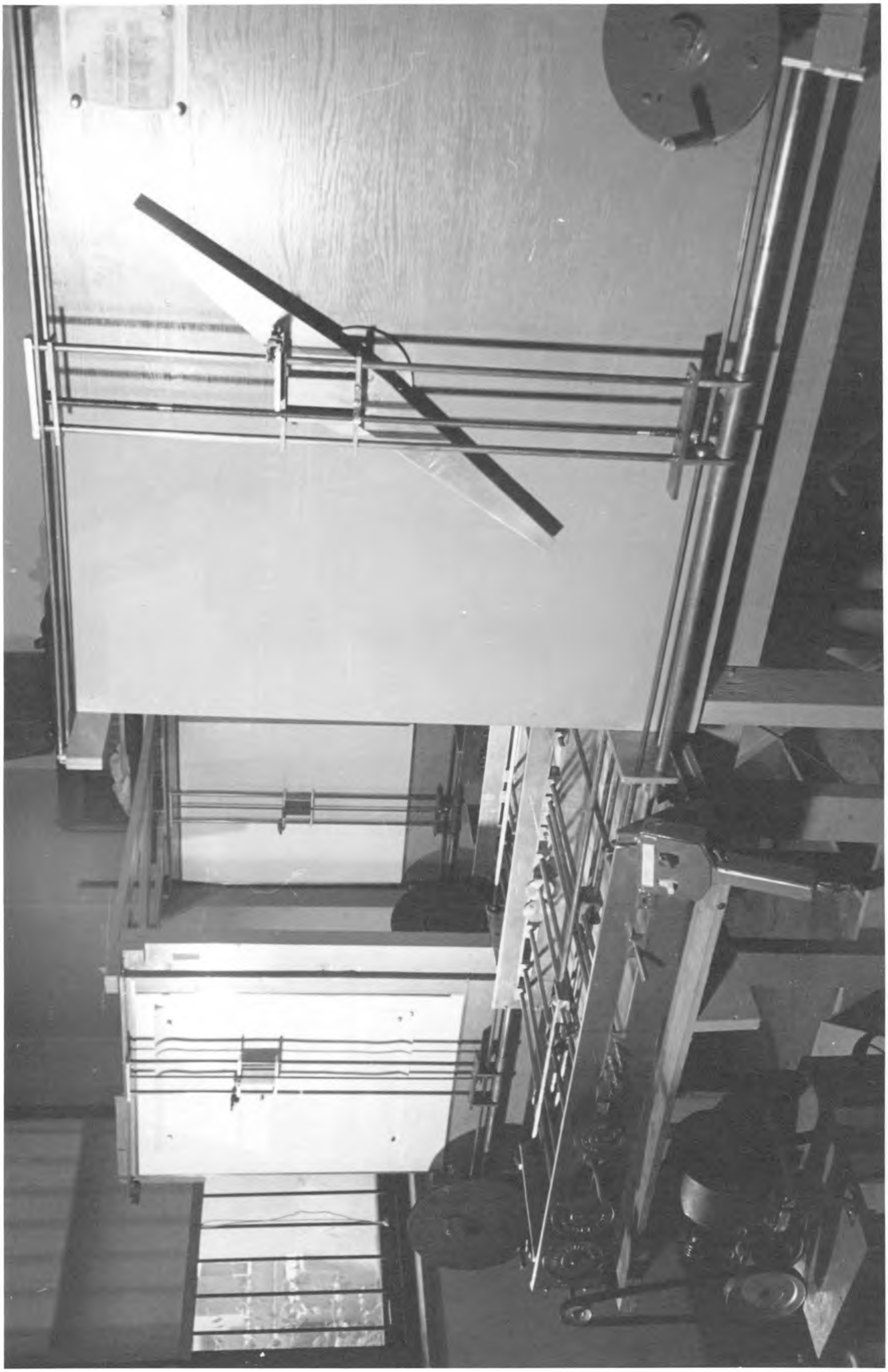

FIG. 16

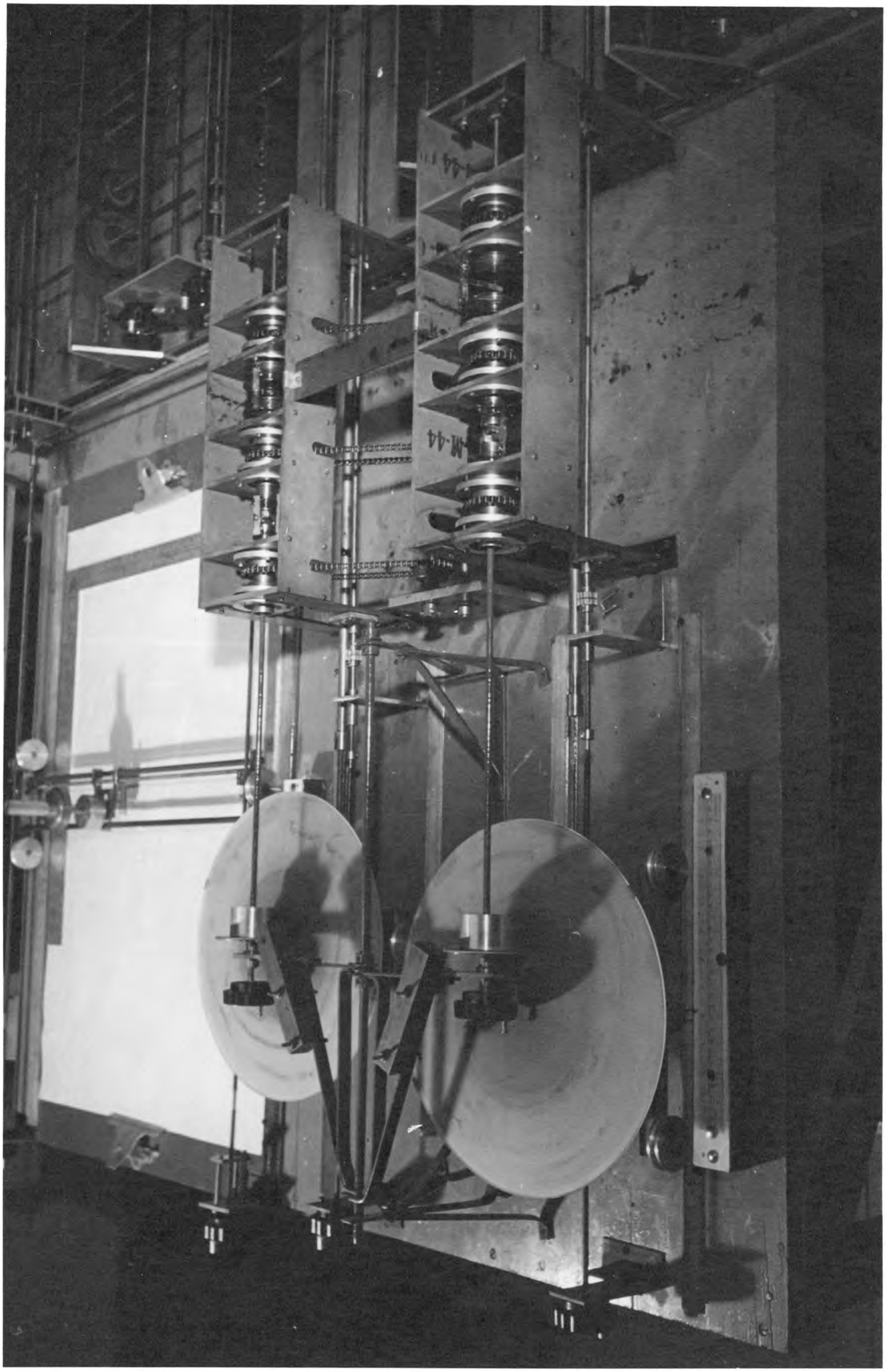

FIG. 17

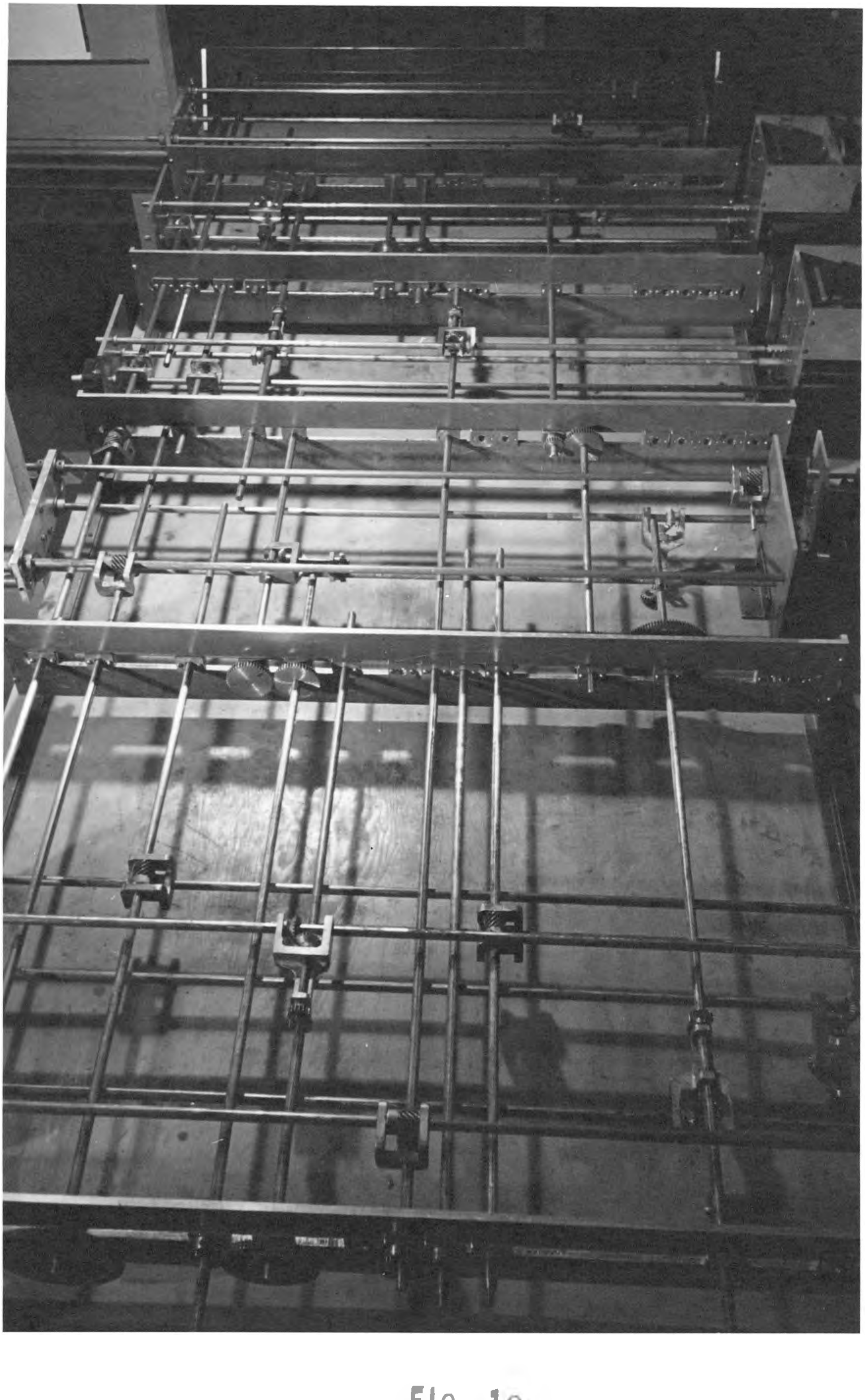

FIG. 18

The first of these shows the integrators and amplifiers clearly enough so that all the essential parts can be made out, and the latter one shows similar detail in the shaft system. It is believed that all the discussion to follow will be quite clear if the reader will turn from figure to figure in his perusal.

Each integrator is made according to the scheme outlined earlier in this paper. In each, the main structure is a box of 1/4 inch aluminum plate riding on three wheels on two rails. The three wheel arrangement was found to be superior to a four wheel one, for in it only the pair of wheels on one side act as guides, and competition between pairs on two sides does not occur. (The third wheel is allowed to have considerable play). Each disc is of glass, roughened by application of fine carborundum. Inside the box, a pair of miter gears makes a right angle connection to a shaft that runs across the shaft system; turning this latter shaft causes the disc to rotate. As the carriage moves laterally back and forth, it is necessary that this shaft alter its length, and this is accomplished in the following way. The cross shaft, where it leaves the shaft system and passes under the amplifiers, is attached to the end of a brass tube; this tube extends some six inches past the edge of the amplifier

nearest the carriage. The shaft that attaches to the miter gear in the carriage box is a square steel shaft, and at its other end projects inside the brass tube, through a square hole in a fitting at the end of the tube. When the cross shaft turns, the square hole at the end of the tube forces the square shaft, and hence the disc, to turn, while as the carriage moves one way or the other, the square shaft slips into or out of the brass tube. Thus the length of the whole shaft assembly is automatically adjusted. A spring arm arrangement at the end of the square shaft inside the tube prevents free oscillatory motions of the end of the square shaft.

The cross shaft in the shaft system which corresponds to the input (integrand) for the integrator runs to about an inch from the edge of the amplifier nearest the carriage. At this point a simple clutch is provided, by means of which this shaft can be coupled readily to the lead screw that runs through a threaded hole in the integrator carriage to a handle at the left edge of the table. This handle is used to set initial conditions, and the clutch is needed in order that the integrator can be disengaged from the main body of the machine while the initial conditions are being set.

Each integrator carriage bears a pointer projecting from its side; this pointer rides just over a scale that parallels the rails on which the carriage travels. The scale is a ruler marked off in tenths of inches, and since the integrator lead screws have 20 turns per inch, each one-tenth inch scale division corresponds to two turns of the lead screw. To set an integrator displacement one reads the position of the pointer to the nearest tenth-division marker (which means to the nearest integral number of turns of the screw), and then introduces the finer part of the setting (fractional number of turns of the screw) by reading the position of a mark on the circular handle by means of which the screw is turned.

The scale is positioned carefully so that the reading is 0.00 when the integrating wheel is at the center of the disc. This "zeroizing" is accomplished by a method explained later in this paper.

The precision with which settings of the integrators can be made has a large influence on the accuracy with which solutions to problems can be produced by the computer, when the problems in question depend at all critically on the initial conditions. This is one matter in which the State University of Iowa computer is a bit deficient; it appears that if an integrator displacement system of very high accuracy is wanted, then the realm of comparatively crude parts must be left.

The settings of the integrators in the State University of Iowa machine can be made to the nearest 1/20 turn of the displacement screw; this is the amount of movement of the screw required to produce a just visible motion of the pointer over the scale, and that this much is needed is due to looseness and play in the system. However, experience has shown that the smallest fine setting that can be made with high reliability is more nearly 1/10 turn. The influence of this factor on overall machine precision will be discussed later in this paper, but it might be mentioned now that the accuracy attainable in making the integrator settings, as the apparatus now exists, is consistent with the sizes of errors introduced by other faults in the machine, such as backlash in the long gear trains.

Each integrator wheel is made of steel, and has a diameter of 32.000 ± .002 inches in order that the integrator constant be 64, as has already been explained. Each wheel is rigidly attached to a shaft that goes into an amplifier unit, and at a point near the wheel itself the shaft turns in a ball bearing which in turn is solidly fixed in one arm of a triangular arrangement of steel bars. The plane of this triangle rotates freely about an axis coinciding with the edge of the triangle opposite the corner where the bearing is located. As a consequence of this freedom, when small bobbing motions of the integrator wheel occur while it is following the rotations of the disc, contact between wheel and disc is not lost. To further insure this, and to prevent slipping, a fairly heavy hub is attached to the wheel, and rather large steel weights, adjustable in position, are mounted on top of the triangle. The integrator wheel shaft is prevented from moving laterally, in response to the dragging force caused by motion of the glass disc back and forth, by retaining collars. Knobs are provided at the ends of the integrator wheel shafts near the wheels themselves; these knobs are useful for manual introduction of backlash compensation, which is a matter that will be discussed later.

The torque amplifiers each consist of two stages, and each pair of stages produces an amplification factor of something like 5000, as was determined to a rough approximation by hanging weights from small lever arms at the input and output of a unit. If called on to produce more amplification, parts of the amplifiers begin to fail; usually the bands begin to pull loose from the binding screws fastening the ends of the bands, and at the same time a great deal of shaking occurs because the bands become too tight to slip freely enough on the drums.

The motor that spins the drums in both amplifier units is under the table that supports the integrators, and a system of gears, sprockets, and chains conveys the rotations of the motor to the drums, each drum getting a rotation in the proper direction. The drums are made to turn at about 400 revolutions per minute, and it might be pointed out here that if one is ever forced so to scale a problem for machine solution that the integrator wheels must turn at a high rate, care must be exercised to avoid having this rate approach 400 per minute, for in such an event the amplifiers will fail to work for obvious reasons.

The drums are of polished steel, the first stage drums being one inch, and the second stage drums two and one quarter inches, in diameter. (The units work very

distinctly better if the second stage drums are quite large.) In each first stage, the bands are of a type of cable intended for use in radio dial connections; it has a core of fibre glass, which does not stretch, and an outer sheath of braided phosphor bronze, which does not scour the steel drums. In each second stage, the bands are of nylon cord intended for use in stringing badminton racquets. At an early stage in the development of the machine, the problem of finding suitable materials for the bands was troublesome, and a rather large variety of materials was tried. The choices named above have been found to be by far the most satisfactory. Lubrication is provided by squirting a bit of machine oil over the bands and drums occasionally; such lubricants as specifically prepared suspensions of graphite in benzene, which can be found recommended in the literature, are certainly not necessary. A bit of care is required to wipe off old oil when it begins to get gummy, as it often does, for this condition causes a jerkiness in operation.

New nylon bands are found to have a "stretch" of about one inch for a three foot length subjected to about 50 pounds of force, but after the amplifiers have been run for a while, this stretch evidently disappears effectively as long as the integrator loads never become

very great.

A word should be said about adjustment of the bands in the amplifiers; this is important, but fortunately does not demand very frequent attention. The computer being described has been run a great many hours since it was put into operation, and has several times been run as much as four hours almost continuously in a single day, and yet it has never been necessary to replace any of the original bands installed in the amplifiers. However, the adjustment of the bands tends to become faulty after every fifteen or twenty hours of operation. This is, of course, an excellent record of reliability and attests to the wearing quality of the types of materials used as bands.

A state of good adjustment of the bands exists when all are equally tight, and it has been found that this sort of balance can be tested for conveniently by spinning the integrator wheel, which must be lifted clear of the glass disc for the purpose. If, when the wheel is released, it continues to make three or four rotations before coming to a stop, and if it does this for spins given it in either direction, then the balance can be considered satisfactory. If some one band is too tight, or if some other unwanted condition exists, this test will

not fail to reveal it. If the bands become maladjusted while the machine is being run the operator will usually be able to detect this condition by a new unsteadiness that appears in the amplifier behavior at once; often this condition gives an audible as well as a visible alarm.

It is possible to have the bands well balanced, but yet too tight. The words "too tight" here mean, not that the amplifiers will not work properly, but that while performing well, they get excessively hot because of the increased friction. Therefore it is necessary to test the heat being generated during operation; if the finger can very easily stand the touch of the drums, they are not too hot. A surprisingly small amount of tightening of the bands will cause the drums to become unbearably hot to the touch.

After some months of use, the chains driving the drums tend to slacken. The whole amplifier units are attached to the supporting table by wood screws in angle irons, and the slack in the chains can be removed by loosening those screws, forcing the units apart a little (not much is required) and inserting spacer bars. Such a bar can be seen in the photographs. Keeping the chains taut should be considered an important part of the maintenance of the computer for slack chains vibrate about

violently, and even a little looseness permits them to jump teeth in the sprockets.

So far the only use of ball bearings mentioned was in connection with the description of the method of supporting the integrator wheel shafts. In addition, all the amplifier drums turn in large ball bearing mounts; use of such bearings in these places is necessary. However, nowhere else in the machine were ball bearings used. In some places in the integrators and amplifiers, oillite bearings were installed, and this kind of bearing was also used in supporting the ends of all the cross shafts in the shaft system. All other shafts in all other units, and all the bus shafts in the shaft system, simply turn in bored holes in the supporting plates. If a shaft is seated squarely in such a hole, and oil is applied, undue frictional drag does not appear.

Little expenditure of words to describe the input and output tables is required. The pointer (or pencil) in each is borne by a small carriage, in which is a threaded hole riding on a lead screw. The right angle connection at the base of each such lead screw to the cross shaft is provided by small brass miter gears. As the lead screw as a whole travels laterally, it is necessary that the miter gear connection be free to move with the screw.

This motion is made possible by having the cross shaft, which is of circular cross section in the part of it that lies in the shaft system, become of square cross section where it parallels the unit over the length across the lead screw must travel. The miter gear mounted on this square shaft is bored with a square hole, so that as the lead screw moves laterally, and the cross shaft referred to is rotated, the miter gear pair slides along the shaft while it follows the rotations. The lateral motion of the lead screw itself is provided simply by another lead screw, turned by a cross shaft in the shaft system and running in a threaded hole in a block at the end of the travelling lead screw. It is believed that the manner in which this can be done is obvious to such a degree that no further explanation is required.

The cranks by means of which the input table operators follow input curves are all double. That is, two handles are provided, one with about twice the length or torque arm of the other. It was found that this availability of a choice of arms is of great advantage to the operator, who cannot maintain smooth motion over slowly changing parts of curves with a small arm, nor maintain sufficient speed over rapidly changing parts with a large arm. It developed that the operator of an

input unit never has to work very hard, even when his cross shaft is coupled to pretty heavy loads, so that sometimes when large scale factors in the set-up design are unavoidable, it is possible to throw them onto the input table operator, so to speak. The large mechanical disadvantage given the operator in the sample set-up given previously is an example of this.

In the output and input tables, wherever the various lead screws turn in threaded holes, the holes are actually half-nuts, each lead screw being kept in contact with the threads of the corresponding half-nut by a bit of clock spring metal. By lifting the spring off the screw with the finger tip and removing the screw from contact with the threads of the nut (the screw is sufficiently flexible for this to be done), the carriage can be quickly and easily set in any desired position. Presence of these devices makes for a great saving of time and avoids the nuisance of having to make hundreds of rotations of the screws to set the carriage positions.

Most lead screws in the machine have their threads cut away for a space of about one half inch at those places that would correspond to excessive displacements of the units driven by those screws. For example, the integrator carriage lead screws are turned down at each end so that

when the carriage is displaced as far as could be without having the integrator wheel drop off the edge of the glass disc, further displacement cannot take place.

The multiplier used in the State University of Iowa computer is of a form never used before, as far as the writer has been able to determine. It is curious that the multipliers incorporated in large differential analyzers have evidently all suffered from defects such as inability to accomodate changes in sign of the multiplicands, or such as being restricted to limited angular movements of the multiplier bar because of the kinds of driving linkages employed. For example, it is common to have the unit made almost exactly as drawn in Figure 5 (a), and it is clear that such a unit cannot work in all four quadrants.

In the State University of Iowa machine, the principle by which the multiplier works (as explained earlier in this paper, is retained, but the physical realization of the unit is so designed that all of the four possible variables that enter into operation of the unit can have either sign, and there are no "blind" or unusable regions in any quadrant. The design was originated by Mr. George Ludwig, of the Physics Department, State University of Iowa.

The reader, by referring to Figures 15 and 16, will see that the multiplier bar now actually consists of two bars, rather than of one in the form of a right angle. One of the bars bears a slot along its length and lies on one side of the vertical plane that supports the unit, and the other bar bears an etched line along its length and lies on the operator's side of the vertical plane. These two bars are at right angles to each other, and adjustments to maintain this relation can be made by loosening or tightening a screw provided for the purpose. A pair of lead screws that supply a carriage with two degrees of freedom is present on each side of the vertical plane; on the machine side of the multiplier the carriage bears a peg that rides in the slot, and on the operator's side the carriage bears a pointer.

Clearly, whatever the positions of the carriages, there will always exist two similar triangles corresponding to those of Figure 5 (b), and the operator, in forming the product, uses an appropriate relation among the sides of the triangles. A little reflection will show, too, that whenever any input variable changes sign -- that is, goes through the center -- the operator will produce the product properly in a continuous fashion

during the change of sign. It should also be noticed that if a simple product is being formed, the quantity x in the relation $y = wz/x$, as given in the discussion of the multiplier previously, will not be zero, and it is impossible for the peg to approach the center point of the slot so closely that any instability or indeterminacy in the orientation of the bar can result. If a quotient is being formed, it is of course necessary to avoid having x become nearly zero anyway, so that no instability can result in this case either.

In order that the peg can ride freely enough in its slot, the peg must be somewhat undersized. The bar would then be able to flop about more than is allowable, but this is avoided by a small weight hung on the bar so that the bar is constrained to rest firmly against one side of the peg at all times. In order to prevent looseness of the peg-carriage on its guide shafts and lead screw from introducing shaking or oscillations into the motion of the bar, a sizable counterweight, visible in Figure 15, is attached to the carriage.

The shaft system, as it appears in the State University of Iowa computer, is certainly also quite unique in design, but it derives its character not from an effort to make it better than those in other differential

analyzers but from a need to find an easily constructed form which would yet be easy to manipulate in setting up problems on the machine. The adopted design has proven itself to be so excellent that it can nevertheless be regarded as indeed representing an improvement over previous designs.

In most large differential analyzers, there are a large number of permanently located bearings throughout the shaft system, all very carefully positioned and aligned so that shafts inserted anywhere will line up with all bearings along its length, and so that when gears are slipped on shafts, the degree of mesh of the teeth will be correct. To make such a system is a shop job of no mean size, and was quite out of the question in the present case. The scheme actually used involved some machining, but this was largely a matter of milling and boring, and did not require any extreme precision nor entail any close alignment work.

The first task was to cut six pieces of 1/4 inch aluminum about five inches by three feet in size, and mill a slot of 3/4 inch width down the length of each. These pieces were then mounted on end supporting plates (in turn fastened to angle irons running the length of the shaft system table) at intervals along the shaft

system, with all the slots lying closely in a single plane, as the figures show. Next, a small bearing unit was made and duplicated some 200 times by a sort of mass production method. This bearing unit consists of a small, approximately square piece of aluminum with a thickness slightly less than 1/4 inch, with a width equal to that of the slots, and with a length approximately the same. One of the broad faces of this little block extends out on either side past the edges which are to fit into the slots. This extension, made by milling parts of the edges of the blocks down, is about 1/16 inch thick and projects about 1/8 inch. Thus, these blocks comprise a little face wider than the slots, with a parallelepipedal projection just wide enough to fit the slots. For each such unit another and separate face of the same dimensions as these just referred to was made from brass strapping. Each block and brass face are held together by two 6-32 screws, and at the center of each such assembly, a 3/8 inch hold is bored, this hole being intended to carry a shaft in the shaft system.

Each such bearing unit is installed by removing the brass face, fitting the aluminum block into a slot, and replacing the brass face. When the screws are tightened down, the extensions on each side of the bearing

unit grasp the large aluminum plates with such force that it is impossible to move it even with sharp hammer blows.

All the shafting required for a machine set-up for solving a problem is put in place by selecting (from a stock of 3/8 shafting of various lengths kept close at hand to the machine) shafts of suitable lengths (which can be judged from the final set-up diagram) and locating at approximately correct positions throughout the system enough bearing blocks to carry all the shafts required. It is next necessary to install spur gears and make all right angle connections to cross shafts, and as this work is carried out, each bus shaft, one by one, will have its exact location determined, and then the bearing block screws can be fastened down. Figure 18 shows a typical machine set-up, and it will be evident how easy it is to put shafts in and locate the bearings wherever needed.

Each spur gear is fixed in position on its shaft by a set screw, and in order that gears can always be slipped on and off easily, even though the set screws raise ridges where they grip the shafts, all shafts, including the cross shafts, are flatted. Occasionally it will happen that the flatted part of a shaft will prove to be in an inconvenient position when gears are to

be tightened down, but in such cases it has been made a practice to tighten the set screw elsewhere than on the flat, and then to remove the burrs caused by this with a file as soon as the gear is removed.

Right angle connections are made by two different types of gears. In an early stage of development of the machine, it was supposed that a good way to make such connections would be to use spiral gear pairs, but it was soon found that such gears alone will not work with shafting as small as 3/8 inch. The reason for this is that the spiral gears, having 45° teeth, attempt to force themselves apart, and if the driving torque is large, the shafts will be made to bend enough to allow the gears to disengage and re-engage violently. It was found that retaining boxes that prevent the shafts from yielding enough for this to happen solved this problem; in Figure 18, such boxes can be seen wherever spiral gears are present. These boxes are made of bronze, and are of obvious design.

At this point, it was found that a further, unsuspected difficulty existed, that made use of miter gears necessary in many places instead of spiral gears. This difficulty is that spiral gears, even when retainer boxes are used, are very wasteful of input torque. Thus, if one shaft of a pair of shafts coupled by spiral gears

is turned, only a fraction of the original torque is available at the end of the second shaft. If even as few as three pairs of spiral gears are used in a train, then so much of the original torque is lost that the efficiency of the chain is very low, and even small loads will make it impossible for a strong man to turn the input shaft. What is wasted goes into end thrust and sliding friction between the teeth.

Such problems do not exist if right angle connections are made by special miter gear boxes; these boxes can be seen in Figure 18. There it will be noted that each unit uses small spur gears as well as miter gears; the spur gears are necessary in order that the motion of the miter gears, which of necessity must lie in one plane, can be conveyed to the lower plane of the cross shafts. These gear boxes work very well, but require considerable work to make, so that it was decided to use spiral gears at some places in the shaft system to reduce the number of miter gear boxes needed, but never to allow more than one pair of spiral gears to enter into the gear train linking any two machine units.

Both spiral gears and the miter gear units just described require that the coupled shafts lie at different elevations. Actually three different shaft levels exist

in the shaft system; all bus shafts lie in the plane determined by the slots supporting the bearing blocks, and some cross shafts lie above, and some below this level, in each case at a distance required by the sizes of gears used.

All right angle connecting units are left permanently on the cross shafts. One reason for this is that, while bus shafts are easily removed and inserted at will, cross shafts are attached to units of the machine and are not easily removed. Another reason is that if a miter gear unit is put on the shafts in one way, it makes a right handed connection, but if put on in another way, it makes a left handed connection. It was thought best to leave the gear units permanently on each cross shaft in order that the sign of the connection that could be made to each such shaft would be permanent and definitely known. Whenever a connection of the opposite type is wanted, a pair of spur gears can be inserted to accomplish this end.

The adder was put together out of two steel gears of 2 inch diameter, and four brass miter gears somewhat smaller in diameter. It is not difficult to make an adder that works very well, but the unit is sensitive to small misalignments in the assembly so that some tinkering is required. The simplicity of the adder makes more description of it than that given earlier in this paper unnecessary.

Some of the dimensions of parts of the computer and some of the lead screw pitches are of importance in set-up design, and for this reason the data of Table I are given here:

TABLE I

Some Significant Dimensions

Input Tables:	Pitch of operator's screw	16 turns/in.
	Pitch of the other screw	20 turns/in.
	Total traversal for input	520 turns
	Total traversal for output	288 turns
Output Table:	Pitch of both screws	20 turns/in.
	Total traversal for first screw	540 turns
	Total traversal for second screw	380 turns
Multiplier:	Pitch of all screws	16 turns/in.
	Length from center to either end of slot or etched line in multiplier bar	184 turns
Integrators:	Pitch of displacement screws	20 turns/in.
	Total traversal of carriage to either side of center of disc	120 turns

It is also useful to have a list of the spur gears available. This information is contained in Table II. All gears listed are of 48 pitch.

TABLE II

Spur Gears Available

Number of Teeth	Number of Gears	Number of Teeth	Number of Gears
16	1	45	2
24	30	48	30
26	2	54	3
28	6	56	3
30	4	60	2
32	4	64	4
36	6	66	3
39	2	70	2
40	4	84	6
42	2	96	4
44	2	144	1

Finally, although it has been given twice already, the value of the integrator constants, which is 64, may be stated here in order to make the list of useful numbers complete.

Chapter IV
TESTING THE COMPUTER

It will have been realized by now that the planning of even a small model differential analyzer involves a great deal of careful attention to details. It can also be imagined that the assembling of the parts requires much fussy work in levelling discs, aligning shafts, and the like. Once the machine is completed there is no way to estimate the accuracy obtainable from the quality of the planning and work done, however, and the accuracy must be learned by specially devised tests. These tests will be described in this chapter, and in the next.

Besides the matter of testing the machine as a whole, there are some allied topics having to do with adjustments of the integrators that can be appropriately discussed here, and these questions will be dealt with first.

In Chapter III, it was mentioned that the displacement of the integrator disc from the center position is read off from a scale. It is essential that the scale reading be known when the point of contact of the wheel with the disc coincides with the center of the disc, and

each integrator must be thus "zeroized". (Actually, to avoid troublesome constants when making settings, the scale is itself adjusted so that the reading is 0.00 when the wheel is at the disc's center.) The zero point is first located pretty well by taking the mean of two readings for which the wheel has small, but perceptible, rotations in opposite senses. Then the reading is made equal to this mean, and the wheel (or rather the knob attached to it) is watched to detect rotation. This is a tiresome process, and is facilitated a good deal by spinning the disc at as high a rate as practicable. When it is certain that the wheel is still rotating, small changes in the setting are made, until eventually a setting can be found such that the wheel is essentially motionless. The meaning of this phrase, in connection with the State University of Iowa computer, is that the wheel makes no more than 5° of rotation for about 600 turns of the disc. Because it is impossible to control the settings themselves with great precision, it is useless to try to improve on this performance, for an extremely accurately located zero cannot be recovered well enough to justify such improvement.

The integrator wheel must also be adjusted in position with respect to side-to-side motion; when the

disc is displaced, the wheel must pass directly over the center of the disc, and not to either side of center. This adjustment is made by means of bolts in slots in the metal arm supporting the ball bearing mount in which the wheel shaft turns. The procedure consists of setting the wheel in one position, moving the disc back and forth under it in the vicinity of the zero position, and watching for rotations of the wheel caused by failure to pass over the exact center of the disc. Actually, this adjustment is not highly critical, for a change of 1/32 inch in the position of the wheel sideways does not seem to cause any sizable rotation of the wheel.

It has been found that the zero points do not remain fixed permanently. Once, the change in the zero was definitely associated with a stress to which the integrator was subjected when a person leaned across it in an effort to reach some other part of the machine; perhaps some settling and wearing of parts in normal use also contributes to alterations in the zero adjustment. In any case, a check must be made occasionally to make sure the setting is still correct. It must also be born in mind that the zero setting is not independent of any adjustments made in the sid-to-side positioning of the wheel; if the latter is ever changed, the former must be

rechecked.

Another matter that must be taken care of is the accurate determination of the integrator constant. This can be done in various ways, but since the State University of Iowa machine was not one capable of the very highest precision, it was not necessary to use any really painstaking method. In the large differential analyzers, an accuracy of 1 part in 5000 in the integrator constant is desirable. A comparable figure for the State University of Iowa machine would be 1 part in about 1000.

It has already been explained how the integrator constant happens to exist; from that explanation it will be clear that if the integrator constant is exactly 64, then when the integrator displacement is made equal to 64, the wheel and disc will make the same number of rotations in equal times. This is the basis of the test used. At one end of the shaft system, there is made to project out an end of a shaft coupled to the independent variable drive shaft. By means of right angle connections, the output of an integrator can be transferred to another shaft which can be made to project from the end of the shaft system alongside the shaft referred to just previously. One then fastens a large spur gear to the projecting end of the ed

of each of these two shafts, and at a point on the edge of each, corresponding to its nearest proximity to the other, a mark can be made. The amplifiers and time drive are then turned on. If the integrator displacement is made equal to 64, and if this is the correct constant, then the two spur gears will turn at the same rate. One lets the machine run for some time, and then checks to see by how much the marks on the gears fail to agree.

In the State University of Iowa machine, when the displacement of the integrator is made 64, only about an 8° failure of the marks to coincide is detected after several hundred turns. However, the value to be attached to such a setting is in doubt by at least 1/20 of a turn of the displacement screw, so this in itself is not a full measure of the accuracy of the integrations performed. It is in fact impossible to attach one single value to this accuracy, because even if the output of the integrator is correct to a certain degree when the displacement is 64, it will not be so correct when the displacement is only of the order of 1 turn, for the relative error in the setting is itself then large.

It is certain that any error in the integrator constant is much less in its effect on the accuracy

of integrations performed by an integrator unit than is the error in the displacement of the integrator. The tests made would lend credence to a value of 64.00 $\pm$ .04 at the very worst imaginable, while the error in setting a displacement of 64 is of the order of 1 part in perhaps 600. One concludes that the constant is 64 to a degree of approximation consistent with the accuracy with which displacements can be made.

With these matters taken care of, an examination of the overall accuracy of several interconnected units of the machine is necessary, and then the question of the accuracy of the entire machine, operating as a single computing instrument in the solution of differential equations, will remain. The accuracy of groups of interconnected units was tested by methods to be described beginning with the next paragraph. The accuracy of the machine as a single ensemble can be judged by a type of test known as the "circle test", but by far the best and most convincing tests are made by solving equations with calculable solutions and comparison of the machine solutions with the known solutions. The circle test and the solution of certain problems will be described in the balance of this chapter, and when this discussion has been completed it will be seen what the general accuracy is. In the

chapter to follow this one, application of the machine to certain other problems will be described, partly to demonstrate the sort of work that can be done with the computer, but also to demonstrate points about the accuracy attainable. One should regard this discussion of the accuracy as only started in this chapter, and this reflects the fact that the accuracy depends somewhat, and sometimes very much, on the problem being solved, so that the question of accuracy attainable must always be faced anew when a differential equation is given to the machine for solution. The accuracy, in short, is a variable quantity depending on the problem being treated, but a pretty good upper bound can be stated which will hold a fairly generally.

It is not proposed to discuss in any detail the tests made of groups of units coupled together, but only to indicate what was done and what results were obtained. The input units were used in conjunction with the output table to see how well an operator could feed an input curve into the machine. The resulting tracings showed two faults; firstly, there tends to be a small waviness in the line, the waves being about the width of a pencil line in size, and, secondly, there occasionally appear deviations from the true curve of about 1 part in 100 or so in size, as a sort of average. The production

of functions of functions by use of two input tables coupled together gives similar, but slightly worse results. The multiplier was tested by feeding into it two input functions from input tables. The product formed again was correct except for the small waviness and occasional deviations of moderate size.

It is claimed that it is not necessary to concern one's self with the problem of reducing operator errors below this size. This claim is supported by the fact, experimetally verified, that such errors as are produced by the operators of machine units get greatly reduced in significance and effectively smoothed out at the integrators. That this should be so can be seen by considering the following argument. Suppose the displacement of an integrator at an instant in time is, say, 100 turns, and suppose that the inputs or multiplier, or both, (in conjunction with an adder, perhaps) are displacing this integrator. Because of the scaling procedure in set-up design, it will always turn out that the integrator wheel must make many turns (perhaps many dozens in extreme cases) while the displacement changes by one turn. It is the case that a careful operator can scarcely wander from his curve by more than one turn for a very short time. Scale factors between the operator's

unit and the integrator displacement screw will usually further reduce this error to a part of a turn. The operator's error, which would appear as perhaps a part in a hundred or so if his output were being recorded directly, now actually affects the total rotations of the integrator wheel by a very small amount. When the integrator displaces from 100 to 101, say, it may make perhaps 30 turns when the displacement is correctly made. But if the operator of an input unit creating this displacement makes an error of even one turn and then returns to his proper position, this wandering will be restricted to a time interval during which the wheel will have made perhaps five or six turns. The reader must only consider the nature of the interplay of these actions to be convinced that the consequent error in the integral produced during the displacement from 100 to 101 will be very small. It is, of course, impossible to assess numerically.

There also enters here the fact that some change in the rotations of the dispalcment screws can occur without effect on the rotations of the wheel. This certainly contributes considerably to the smoothing effect on operator errors.

To check all this rationalization, at a later stage of the testing procedure operators were instructed

to introduce very large errors into their work deliberately, and it was found that in normal operation these errors cannot be associated with any faults in the machine solution of a problem. This sort of thing was investigated at considerable length, and the findings are definite and sure: machine solution of an equation is quite insensitive to operator errors, provided the operators are not wholly inattentive, and provided there does not exist some special peculiarity in the problem being treated or in the set-up design being used.

It can be deduced that this result was known to the makers of the large differential analyzers in existence, for even though these are expensive and carefully made machines, none, as far as the writer knows, has used any automatic curve followers or has shown any difficulty connected with errors from input tables to exist.

It might be expected that backlash in the long gear trains used in problem solutions would be the largest source of error present, once it has been shown that operator errors are small. This is indeed the case, and it is in the investigation of this error that the circle test is used. As far as backlash is concerned, errors occur only when the integrator output shaft rotations change signs, for it is only then that the

otherwise steady pressure of gear tooth on gear tooth throughout the chain is relieved, and that some rotation can be lost before all gear teeth again come into firm contact. The circle test is specially designed to reveal these errors.

The machine is set up to solve the equation whose solution is a sine wave, and the inputs to the output table are made to be the solution and its first derivative. This plotting of a sine curve against a cosine curve would result in the drawing of a circle if the machine operated perfectly. But in general it is found that the curve is a spiral rather than a closed curve, and also flat regions may be seen in the spiral. Such departure from a true circle is due to the combined effect of backlash in the amplifier bands, looseness in the fit of lead screws riding in tapped holes, and backlash in the gears in the interconnecting system; the gear backlash is much the largest part of the effect.

The circle test was performed many times, and in addition the sine wave solution was plotted as a function of the independent variable. With the latter kind of recording, errors show up as flats at the peaks of the curves; these flats correspond to the passage of one or the other integrator through its zero. With this type of

recording the first peak was found to be in error by about 1 part in 70 and if many cycles are run off, the error is found to be cumulative, adding about the same amount at each successive peak. The cumulative aspect of the error corresponds to the spiralling of the curve in the circle test.

The error, 1 part in 70, just mentioned, was the difference between the value of the mean curve of a set of eight at the flat and the true value of the sine curve at its peak. Successive runs of the machine solution showed a mean dispersion about their mean curve (at its peaks) of 0.4%, nearly, and the maximum variation found in about a dozen peaks from the mean, was about 0.7 %

It can be suggested that the error, about 1 1/2% per half-cycle, is already indicative of overall machine accuracy of perhaps better than 1%, for the reason that the large number of reversals and passages through zero involved in the circle test constitutes a much more stringent test than is met in most typical problems.

In the large differential analyzers the backlash problem is partially solved when great precision is wanted by the insertion of special compensating devices sometimes called frontlash units. These units typically involve a set of planetary gears so arranged in conjunction

with a friction band and limiting pegs that when the shaft on which the friction band runs changes direction of rotation, the gears come into play and quickly introduce a rather sudden turn of a size that can be prearranged by fixing the positions of the limiting pegs suitably. This sudden turn has the effect of taking up the play in the gear train in a short time. After this action has taken place, the frontlash unit goes out of operation until the next change of direction of rotation occurs.

In the case of the State University of Iowa computer, it has been made a practice to compensate for backlash during reversals of sign by manual introduction of an amount of frontlash. This is done only when a problem being treated involves several changes of sign and the solution is wanted quite accurately. The operator stands by the integrators, and whenever either wheel passes over the center of the disc he grasps the knob provided for this purpose at the end of the integrator wheel shaft and quickly turns it to take up the slack in the gear train. The amount of turn to be made is predetermined by experimentation with the circle test arrangement. It should be pointed out that mechanical frontlash units of the type described above are not suitable for use in the State University of Iowa machine because their functioning

depends on their not being required to drive heavy loads, while in the State University of Iowa computer very considerable driving torques are needed to turn any shafts in it as a consequence of the absence of ball bearing mounts and of the general comparative crudeness of the entire construction.

It has been found by numerous trials that manual introduction of two thirds of a turn of the integreotr wheel shaft at crossings of the zero points is just enough to cause the circle to fail to close by about two pencil line widths, while the circle fails to close by eight line widths when no compensation is used. If more frontlash is introduced in an effort to make the circle close completely it is found that abrupt changes in the length of the radius vector appear. These are only about the size of the width of a pencil line or two, but since they appear as distinct steps in the circumference, it is supposed that two thirds of a turn is full compensation for backlash and more introduces its own error. It should be added that flats in the circle test are just detectable. All in all, it is not considered worthwhile to attempt to improve on this situation for reasons given in the next paragraph.

Persons who have had experience with large differential analyzers have been able to estimate that

when no flats are visible in the circle test and the spiral effect is about half the width of a pencil line or so, the overall error is of the order of 1 part in 800 or a bit worse.[11] In the case of the State University of Iowa computer, it will be seen that when backlash errors do not come into play, overall machine accuracy is of the order of 1 part in 150, so that getting the circle to close to about two pencil line widths suggests a degree of accuracy compatible with the overall accuracy to be expected in general operation. More important is the fact that to make the circle close more nearly consistently would require very extensive rebuilding of parts of the machine.

The sine wave solution was on several occasions run four to six times successively with backlash compensation. The error at all peaks of the mean curves as compared with the true value was uniformly 3 to 4 parts out of 480, or 0.6 to 0.7%. The mean dispersion about the average for a total of sixteen peaks checked was 4 parts out of 480, or 0.7%. The maximum deviation was about 1.0%.

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11. Crank, J. The Differential Analyzer. London, 1947. p. 135.

Comparison of these figures with those given for curves corresponding to no use of backlash compensation will show that while the mean curve agrees with the true curve about twice as well now, both the mean dispersion and maximum dispersion are higher. This could be due partly to inconsistencies in the amounts of frontlash introduced and in the lengths of time taken by the hand and wrist in making the compensation.

There is also considerable difficulty in making the measurements on the curves well enough to be able to deal reliably with quantities of the size of 1 part in 400 or so, when the quantity 400 corresponds to about six inches and when the pencil line forming the curve is nebulous to the extent of one part. Also, the machine has idiosyncracies that are not perfectly uniform in time. It is simply impossible to account fully for such results. It is the case that all efforts to pin down an accuracy value extremely finely are frustrated. When errors of the order of 1/2 or so are found, this figure is itself always a little doubtful, and to trace the source of the error to specific causes is impossible; the machine is too extensive an intertwined network of interdependent parts.

It is felt that what can most reasonably be done to determine what sort of accuracy rating should be given

the machine is to make tests of the kinds described, and of the kinds yet to be described, to accept the findings from each test without trying to "explain" them too fully, and then, as the number of pieces of information increases, to take the worst cases found as being conservatively safe indications of the behavior of the machine.

Such an estimate will be safe, but there is this objection to be raised: It may happen that a solution is wanted with good precision, and that the problem is such that if sufficient care is used in applying the machine to it, the precision could be high. In such a case one would not want to use the overly large error estimate arrived at in the manner stated above. On the other hand one would be at a loss to be able to state what the accuracy is, if no part of the true solution is known for comparison.

The writer suggests that such a state of affairs can nearly always be dealt with by finding, or constructing, another equation that very nearly resembles the one of interest, in that it uses the same machine units, is monotonic or oscillatory in the same way as the original one, depends on initial conditions in a similar way, and so on, and which can be solved analytically. The accuracy obtained when the machine is used to solve this second equation can

then be taken to be a reliable indication of the accuracy with which the machine can solve the original one.

As a next step in the testing procedure, the machine was set up for solution of the well-known equation whose solution represents the motion of a damped harmonic oscillator. This is a step toward solution of more complicated equations, for the shaft system becomes more involved for this equation and an adder must be used. The solutions were run many times, both with and without backlash compensation, and also with a reversl of the proper sign to make the solution non-oscillatory, in order that the accuracy in the latter case could be compared with the accuracy in the oscillatory case.

In brief, it was found that the error in the average curve with no backlash compensation was again about 1.4% per half-cycle in the oscillatory case and about 0.6 % when backlash compensation was used. For the non-oscillatory case the error was about 0.8%. Again, little "explanation" of these figures is possible, but the important fact is that all such figures found so far agree reasonably nearly. One begins to suspect that in general, typical operation, one will always get errors running from about 0.5 to about 1.5%, depending on the problem.

It was intended next to solve equations requiring

use of input units and the multiplier and having known solutions, in order to determine the accuracy in full operation. At the same time, it was desired to turn the machine to the rocket flight problems it was originally built to handle, as soon as possible. In the case of one of those problems, part of the solution was very well known, so that it could serve in making the desired accuracy check. In the case of the other problem, too, some data was at hand to make a kind of check possible. Furthermore, it was recalled that when the computer project was started, it was decided that an accuracy of 2% would be considered successful, and tests already completed indicated that this figure would certainly be considerably surpassed, even when all parts of the machine were employed in problem solution. Because of all these considerations, it was decided that the testing could be continued very well, time could be saved, and solutions for the equations of practical interest could be obtained, by turning the machine at once to application to those problems.

This chapter may well be concluded with a summary of results so far obtained. It might be said that the testing was extensive and indicated that a fair estimate of the accuracy of the machine when no human

operators take part in its functioning would be that it is probably generally a little better than 1%, except that when backlash errors enter significantly the (cumulative) error may become about 1-1/2% for each part of the solution which contains contributions from the backlash. It was also known to be very probable that these figures would not change much when human operators did take part in machine solution, and as will be seen, this expectation was justified.

Chapter V

APPLICATION TO SELECTED PROBLEMS

(1) General Remarks

This chapter will be devoted to a discussion of the use of the differential analyzer in the solution of two problems. The problems in question are second order ordinary differential equations describing aspects of the flight of a rocket, and are of great physical interest. However, it is to be understood that they will be here regarded not as physical problems, but as mathematical problems. This attitude will be taken because the inclusion of a discussion of the solutions in this paper at all can only be proper if they illustrate matters of interest in connection with the application of the differential analyzer, or cast light on the precision with which the machine functions. Nevertheless, a short digression will be made initially to sketch briefly the manner in which the equations to be discussed arise, in order that they may seem not completely devoid of physical content.

The principal idea behind the balloon-launched rocket scheme[12] is that a small, comparatively inexpensive

12. Van Allen, J. A., and Gottlieb, M. B., in Rocket Exploration of the Upper Atmosphere, edited by R. L. F. Boyd and M. J. Seaton, London, 1954, p. 53.

rocket can be used to attain high altitudes by lifting the rocket, before ignition of the rocket fuel, to a height sufficiently great so that aerodynamic drag on the rocket will be small when the rocket is allowed to fire and propel itself upwards. Assuming vertical flight, the one-dimensional equation of motion for the rocket after ignition is

$$m \frac{d^2 x}{dt^2} = T - mg - \frac{1}{2} \rho \left(\frac{dx}{dt}\right)^2 C_D A . \qquad (1)$$

Here t is the time, m is the rocket's mass and is a function of time during burning of the fuel, T is the thrust provided by the jet, g is the acceleration of gravity, ρ is the density of the atmosphere and is a function of the altitude, which is represented by x, C_D is the drag coefficient appropriate for the transverse cross-sectional area of the rocket and is a dimensionless function of the Mach number, and A is the cross-sectional area.

While the balloon is lifting the rocket, the rocket is suspended beneath the balloon on a long line. When the propellant is ignited, the rocket slips out of its suspension, and it is of interest to know whether the subsequent direction of motion of the rocket will approximate the direction of its axis before firing.

This problem can be treated as follows:[13] There are always jet forces on the rocket with non-zero moments about the center of mass, due for example to misalignment of the jet stream with the rocket axis. There thus arises an overturning moment L, and due to this moment the rocket begins to rotate about a transverse axis with uniform angular acceleration. However, as the linear velocity increases, aerodynamic forces come into play and a restoring moment develops. The problem is now simplified by assuming that the center of mass is constrained to move in a straight line, and by supposing that the rocket does not spin about its longitudinal axis. The approximate equation for the angle of attack θ, which is the angle between the tangent to the center of mass trajectory and the longitudinal axis of the rocket is then

$$I \frac{d^2\theta}{dt^2} = T \cdot \alpha \cdot s - \frac{1}{2}\rho \left(\frac{dx}{dt}\right)^2 C_M A\, d \ .$$

Here I is the instantaneous moment of inertia about a transverse axis through the center of mass, C_M is the aerodynamic moment coefficient about the center of mass and is a function of θ and of the Mach number, d is the diameter of

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13. Van Allen and Gottlieb, op. cit., p. 56.

the rocket, s is the distance between the center of mass and the exit plane of the jet, and α is the effective misalignment angle of jet thrust. The quantity α is such that $T\alpha s = L$.

For a simple treatment, I, T, ρ, s, and α are taken to be constant; this can be expected to be fairly good for the early part of the burning period. Also, dx/dt is taken as proportional to t, and C_M is taken to be a constant times θ. The equation then reduces to the following one, wherein a and b are numerical constants:

$$\frac{d^2\theta}{dt^2} = a - bt^2\theta \ . \qquad (2)$$

It is equations (1) and (2) that are to be solved. In the first case, both the altitude and velocity as functions of time are wanted for each of a set of selected initial altitudes, and in the second case the variable θ as a function of time is wanted for various combinations of values of a and b.

(2) Solution of the First Rocket Equation

Inspection of Equation 1 would suggest that a sizable number of input units and, evidently, two adders, in addition to two multipliers, would have to be used in

the machine set-up. However, if one rearranges the equation as written below, some simplification results and the general scheme of the machine set-up becomes obvious:

$$\frac{d^2x}{dt^2} = \left\{\frac{T}{m} - g\right\} - \left\{\rho\left[\frac{A}{2}\,C_D\left(\frac{dx}{dt}\right)^2\right]\frac{1}{m}\right\} \qquad (3)$$

There are now only two terms on the right (those in curly brackets), so that only one adder will be needed. The entire first term is a single function of time, and needs use of only one input table. The bracketed expression inside the second term is entirely a function of the velocity, the density is a function of altitude, and the mass as a function of time. Generation of these three functions will require use of the other three input tables, and formation of the simultaneous product and quotient will require use of the multiplier.

On the basis of Equation 3, one can proceed immediately to begin the design of the machine set-up, but before all the scale factors can be determined it must be known what the input functions are, and something must be known about the ranges of the variables during time of solution. The latter information can be estimated by crude calculations once the former information is at hand,

although in actual fact sufficient information was available from general knowledge of the performance characteristics of the rockets in question so that good guesses could be made concerning the maxima of the altitude, velocity, acceleration and flight time.

The thrust function, T, was taken as constant during the burning time (2.82 seconds) and zero thereafter. Solutions for the equation were to be gotten for launching altitudes of 4,000, 30,000, 50,000, 70,000, and 90,000 feet. The corresponding values of the thrust during burning were to be 6025, 6285, 6353, 6386, and 6399 pounds. The mass, as a function of time, was to be given by (6.14 - 1.08t) slugs for t less than 2.82 seconds, and as constant at 3.09 slugs, thereafter. Variation of g with altitude was to be neglected. With this information, the first term on the right in Equation 3 can be constructed as can the function m needed in the second term.

The density function was taken from the data assembled by the Rocket Panel.[14]

The function $C_D(dx/dt)^2$ was deduced by Mr. E. C. Ray[15] from observational data obtained at the White Sands

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14. The Rocket Panel, <u>Phys. Rev.</u> 88 (1952), 1027.
15. Private communication from E. C. Ray, Physics Department, University of Iowa, October 28, 1952.

Proving Ground.[16] Since its exact form is not of interest in the kind of discussion being given here, it will only be said that the curve that was actually used is closely approximated by the line 2690(dx/dt - 692) $\text{feet}^2/\text{seconds}^2$ for dx/dt greater than 692 feet/second, and by zero otherwise. The value of A was 0.231 feet^2. The whole quantity $AC_D(dx/dt)^2/2$ will be referred to as D hereafter, for convenience.

With knowledge of the density and D functions, the remaining two of the four input curves could be drawn, and all scale factors in the set-up determined. The set-up finally arrived at is shown in Figure 19. This diagram is of interest as illustrating a typical set-up for a rather involved problem and also because it shows how a problem entailing use of input curves in which the values of the functions are vastly different can yet be scaled in such a way that no large gear ratios appear anywhere. An example of what is meant is the following: The density at about 131,000 feet has the value of only about 0.000008 slugs/foot^3, while the D function is numerically some 10^{10} times larger for certain velocities. Nevertheless,

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16. White Sands Proving Ground Advance Data Report Number 221, 6 August 1952.

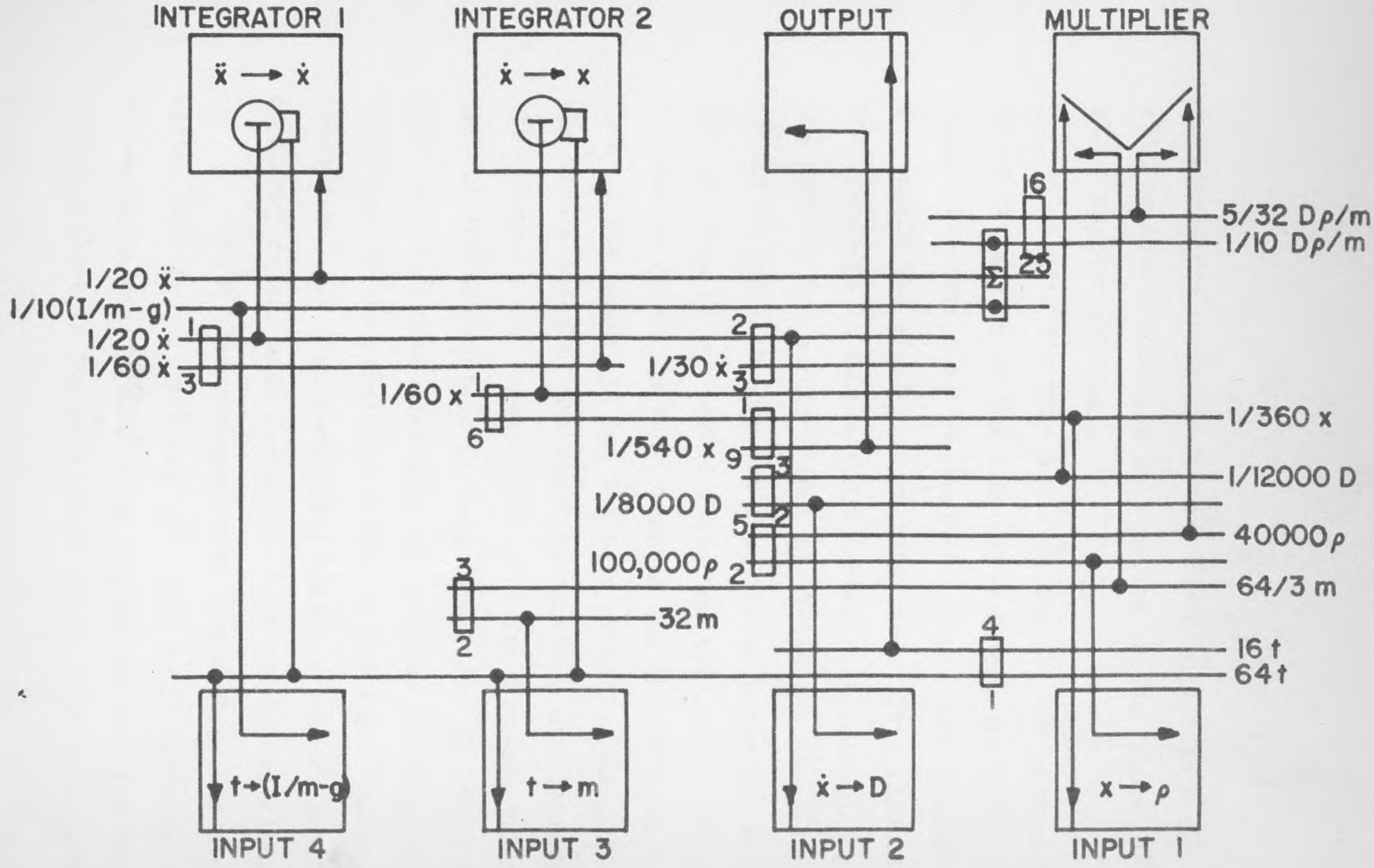
INTEGRATOR 1
$\ddot{x} \to \dot{x}$
INTEGRATOR 2
$\dot{x} \to x$
OUTPUT
MULTIPLIER
16
25
Σ
5/32 Dρ/m
1/10 Dρ/m
1/20 $\ddot{x}$
1/10(I/m-g)
1/20 $\dot{x}$
1/60 $\dot{x}$
1/30 $\dot{x}$
1/60 x
1/540 x
1/8000 D
100,000ρ
1/360 x
1/12000 D
40000ρ
64/3 m
32 m
16 t
64 t
INPUT 4
t→(I/m-g)
INPUT 3
t→m
INPUT 2
$\dot{x}$→D
INPUT 1
x→ρ

FIG. 19.

judicious use of scale factors has resulted in a set-up with no such outlandish gear ratios as the figure 10^{10} might suggest at first.

With the time scale factor 64, and with moderate speed of running of the time drive motor, the solution could be carried to the burn-out point, 2.82 seconds, in roughly two minutes of running time. This allowed the operators to follow their curves with ease during the part of the solution that simulated the powered part of the flight. At burn-out time the computer was stopped and the step in the thrust curve was traversed. It can be seen in Figure 19 that cranking the thrust input table down to the value -32 has the sole effect of changing the displacement of the first integrator, thus introducing the change of the acceleration from its burn-out value to that of the second term at that time, less 32. This input table was then decoupled from the machine to represent the constancy of the acceleration due to gravity. The mass input table was also decoupled; this corresponds to the constancy of the mass during the rest of the solution.

With these changes made, the machine was started again, and the solution was continued until the drag term became quite negligible. This was judged from the position of the pointer in the multiplier unit. When this pointer

was less than one turn of its driving lead screw from its zero position, the operator was in every case moving the pointer exceedingly slowly -- as little as 1/10 turn per minute of running time -- and this is definitely a negligibly small contribution to the displacement of the first integrator. The solution from this point on to the peak, and beyond, can be carried out by pencil and paper calculation, and was not obtained by use of the machine. A valid reason for not doing so was that the time scale required for convenient solution during burning time would have entailed very long runs for each trajectory if the solution were to be obtained to peak altitude. If such solutions were at all necessary, it would be best to rescale the whole problem and reset-up the machine.

For the 4,000, 30,000, 50,000, 70,000, and 90,000 foot launchings the times of flight required for the drag to become negligible by the criterion stated above were, respectively, about 25, 20, 13-1/2, 8-1/2, and 6 seconds.

It will be noticed in Figure 19 that there are two connections between the machine proper and the output table. These two connections were not used simultaneously, but rather one was used while the other was decoupled by slipping a spur gear out of position, and then in a subsequent run of the machine, the other connection was used.

Thus the altitude was recorded during one run, and the velocity during the next. (Clearly, there exists a need for a double carriage on the output table so that two quantities of interest can be recorded simultaneously.)

In the case of the 4,000 foot launching, experimental data were available for checking the machine solution. These data were obtained by the White Sands Proving Ground[17] by tracking the flight of a rocket of the kind in question by two cine-theodolites. The available data included the three components of the velocity and position vectors as functions of time. After applying a correction factor (which was always quite nearly unity) to allow for the fact that the White Sands flight was not vertical, these data could be used to check the machine calculation of the velocity and altitude.

Figure 20 (a) gives the altitude curve as calculated with the mass function stated earlier in this section. Figure 20 (b) gives the altitude curve when the mass function was taken to be a line of the same slope as the one referred to above, but with a burned-out value of 2.69 slugs. Figures 21 (a) and 21 (b) give the two velocity curves corresponding to the altitude curves of

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17. White Sands Proving Ground, op. cit.

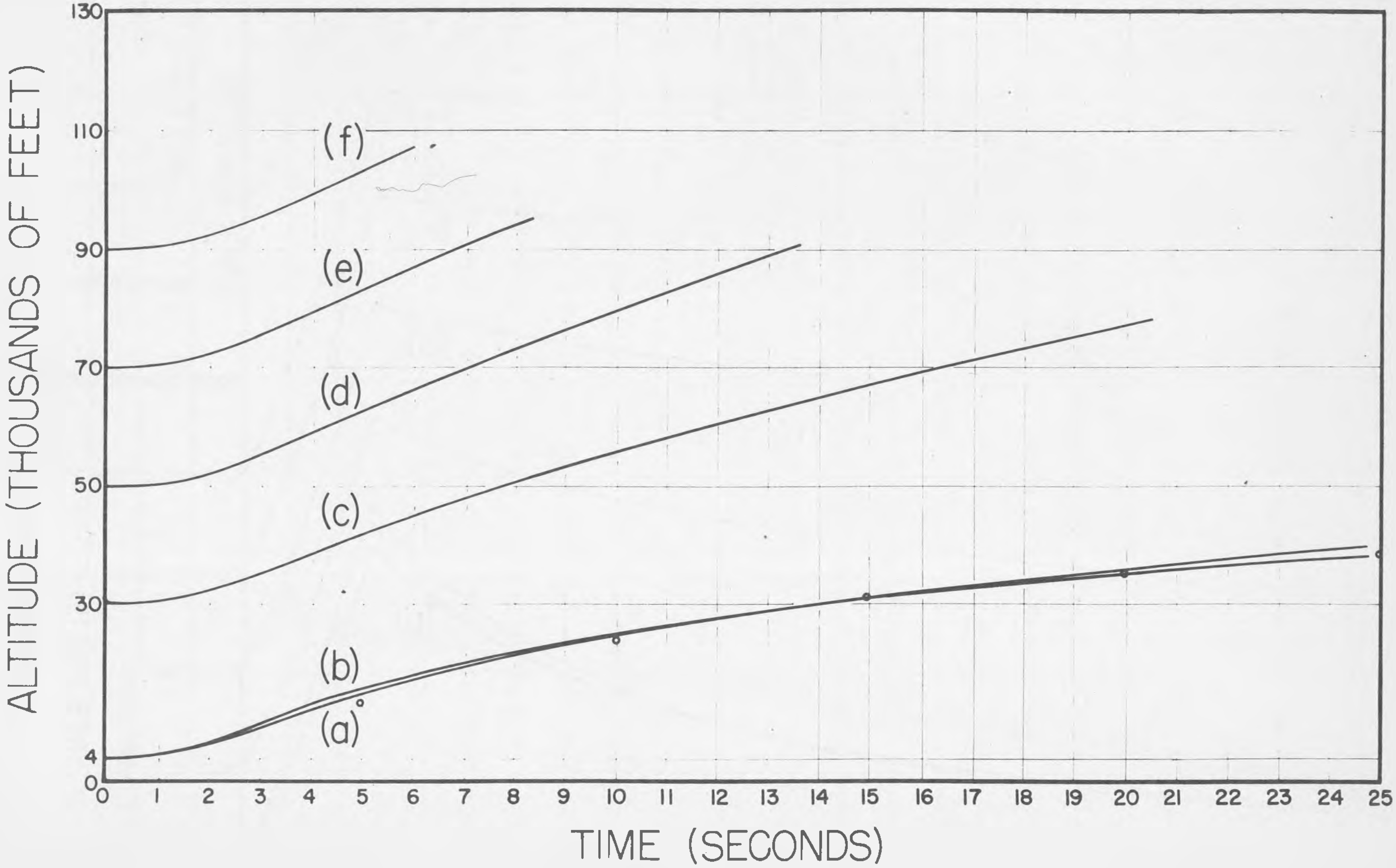
130
110
90
70
50
30
4
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
ALTITUDE (THOUSANDS OF FEET)
TIME (SECONDS)
(a)
(b)
(c)
(d)
(e)
(f)

FIG. 20.

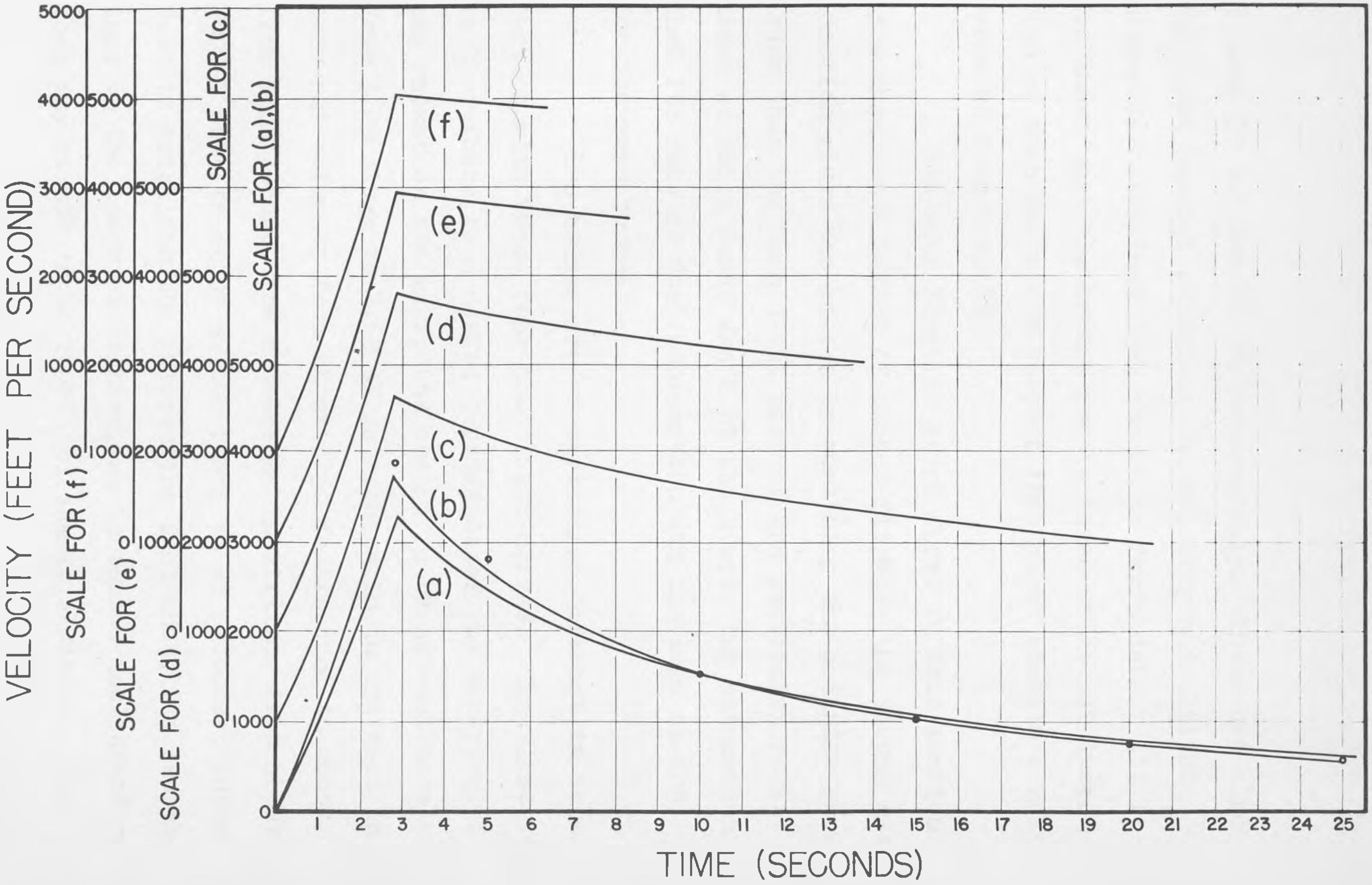
VELOCITY (FEET PER SECOND)
SCALE FOR (f)
SCALE FOR (e)
SCALE FOR (d)
SCALE FOR (c)
SCALE FOR (a),(b)
(a)
(b)
(c)
(d)
(e)
(f)
TIME (SECONDS)

FIG. 21.

Figures 20 (a) and 20 (b) respectively. There are also indicated several points on the velocity and altitude curves as determined from the White Sands data. (It is not known what the errors are in these data, but inspection of them makes one suspect they might contain a mean error of from 2 to 5%.)

The mass function given first in this section is a mean for a number of rocket flights; the second mass function given was gotten by combining the definite knowledge that the burned-out mass of the particular rocket flown at White Sands was 2.69 slugs with the assumption that its rate of fuel consumption was the same as that for the mean rocket.

Two things are of particular interest in connection with these four calculated curves. The first item is that there is a marked dependence of the behavior of the rocket on the mass; the change in burned-out mass from 3.09 slugs to 2.69 slugs results in an increase in burn-out velocity from about 3250 feet/second to about 3750 feet/second, and a decrease in altitude from about 39,900 feet to about 37,800 feet, at 25 seconds. These results establish the interesting fact that, for launchings in the lower atmosphere, an increased instrument payload may result in an increased summit altitude. The

second item of interest is that the machine-produced curves (with the lower mass value) for the 4,000 foot launching agree quite well with the actual behavior of the rocket, but, when all circumstances are considered, as will be done in the following paragraph, not too much faith in such agreement can be maintained.

It has already been said that the treatment of the two rocket problems are included in the present paper to illustrate use of the machine and to extract information concerning the accuracy of the machine. However, while the trajectory problem is a good illustrative one, it is difficult to gauge machine precision by its use. (Quite the reverse situation will prevail when the solution of the second rocket problem is discussed in the next section of this chapter.) Although an observed trajectory is available for comparison with one of the computed trajectories, any agreement obtained might be only fortuitous to an undetermined extent because the drag function is not actually known very well, the mass function used may have been in error, and actual moderately gentle tailing-off of the thrust curve near the end of the burning period was ignored, and other errors in the input curves used certainly existed. Furthermore, as already said, the observational data may be in error by as much as 5%.

It can in any event be said that no reason appeared to cause any change to be made in the accuracy rating previously given. As will be seen in the next section, treatment of the second rocket equation showed that that rating need not be altered, and if it is utilized here, then the calculated curves can be expected to be good to about 1%, although they may be rather better.

Figures 20 (c), (d), (e), and (f) give the computed altitude curves, and Figures 21 (c), (d), (e), and (f) give the computed velocity curves, for launching altitudes of 30,000, 50,000, 70,000 and 90,000 feet, respectively. In all these cases the larger of the mentioned burned-out mass values was used.

On the basis of what has already been said, it will be recognized that it is not possible to say with any certainty to what extent these curves might be expected to represent the actual behavior of rockets launched at t the corresponding altitudes. To answer such a question, one would need to know, not so much what the machine precision is, but rather how realistic the thrust, drag, and other functions used in the solutions are. One might, however, reasonably expect that the computed curves would agree with actual performance more or less as well as the curves for the 4,000 foot launching agree with the White

Sands observations.

It is possible to compare the computed curves with actual rocket behavior to some small extent, for rockets of the kind in question here have been fired from various launching altitudes, and the peak altitudes attained have been estimated from the cosmic ray counting rates reported by the apparatus carried by these rockets. In Table III there are presented the calculated peak altitudes and corresponding times for each of the six trajectories started by the differential analyzer. To calculate these peak altitudes and times, the times, velocities, and altitudes at points at, or quite near, the end-points of the machine-produced curves were read off, and then the equations

$$v = v_o - gt \qquad \text{and} \qquad x = x_o + v_o t - \frac{1}{2} gt^2$$

were employed, the first to determine the time of arrival at peak, and the second to determine the altitude at that time.

TABLE III

Calculated Peak Times and Altitudes

A	B	C	D	E	F	G
4000[1]	24	600	37420	32.13	42.7	43000
4000[2]	24	650	38880	32.13	44.2	45470
30000	20	2020	86490	32.24	82.7	149770
50000	14	2980	92170	31.64	108.2	232500
70000	8	3640	93920	31.49	123.6	303730
90000	6	3830	137030	31.20	128.8	372110

1. 2.69 slugs burned-out mass
2. 3.09 slugs burned-out mass

KEY:

A: Launching altitude (feet).

B: Time (seconds) at which the data under C and D were taken from machine solutions.

C: Velocity (feet/seconds) from machine solution at time shown under B.

D: Altitude (feet) from machine solution at time shown under B.

E: Value of g used in calculation of peak altitude.

F: Calculated peak time (seconds after launching).

G: Calculated peak altitude (feet).

The value of g used in the first two cases was the sea level value appropriate for the approximate latitude of White Sands, New Mexico; the values of g used in the remaining cases are altitude-corrected values, taking the sea level value at the North Pole as base. To get these values, a crude calculation was used to get the approximate peak altitude, and then the value of g for that altitude was found, assuming zero latitude, and this value was used in a new calculation of the peak altitude. In effect, instead of employing a correct, variable g, a constant value corresponding roughly to the peak altitude was used; it was felt that such an approximation was adequate for present purposes. (It does not affect the peak altitudes found much.)

That the values of g for various altitudes over the North Pole were used was due to the fact that the flights, the peak altitudes for which are to be compared with the calculated ones, were made in the far north (at about 77° latitude). The peak altitudes for launching altitudes of 36,000, 41,000, and 57,000 feet, as reported by Van Allen and Gottlieb[18] were 195,000, 210,000, and 295,000 feet respectively. The peak for the White Sands

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18. Van Allen and Gottlieb, op. cit.

flight was 44,400 feet.

One sees that all the calculated peak altitudes are too low, and in some cases by one part in 10 or so. However, it must be recognized that the peaks for the actual flights are only estimates, that the masses of the rockets flown did not necessarily agree very closely with the mass used in the machine calculations, and that many other factors must be considered in comparing the above figures with the tabulated ones. All in all, one can only say that the calculation of highly reliable trajectories requires much more reliable input information than was used in the present computations.

(3) Solution of the Second Rocket Equation

Before design of the machine set-up for the solution of Equation 2 was begun, it was noticed[19] that the equation can be rendered independent of the two parameters a and b by the substitutions $t = b^{1/4}\, T$ and $\theta = ab^{-\frac{1}{2}}\, \beta$. After these substitutions, the equation is

$$\frac{d^2\beta}{dT^2} = 1 - \beta\, T^2 \, . \qquad (4)$$

19. Private communication from Prof. J. A. Jacobs, Physics Department, University of Iowa, Nov. 1, 1954.

It is then necessary to obtain only one solution, rather than a family corresponding to various values of the parameters.

A series solution good to very much better than 1% up to a value of 2 for T was calculated. This range of the variable includes the first peak of the solution, which was known to be oscillatory in nature. Also, a crude solution of this equation obtained with a BEAC[20] (electronic analog computer) indicated that no subsequent crests in the oscillations were as large as the first, and that the frequency increased with the number of cycles. With this information, sufficiently good guesses could be made concerning the maxima of β and its first two derivatives.

The series solution in question was provided by Dr. J. A. Van Allen[21] as were the numerical calculations made with its use. The series can be obtained by assuming

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20. The BEAC was generously made available to the writer by arrangement with Dr. J. A. Van Allen, Head, Physics Department, University of Iowa, by the Applied Physics Laboratory, Silver Spring, Maryland.
21. Private communication from Dr. J. A. Van Allen, Head, Physics Department, University of Iowa, Oct. 20, 1954.

$$\beta = \sum_{n=0}^{\infty} C_n T^n$$

to be a solution. Substitution in Equation 4, use of the derivative of the assumed solution, and consideration of the initial conditions produce the result

$$\beta = \frac{1}{2} T^2 - \frac{1}{60} T^6 + \frac{1}{5400} T^{10} - \frac{1}{982000} T^{14} + \ldots .$$

Table IV presents the set of calculated points used in the accuracy checks on machine solutions. For later reference, there are included in the table the corresponding values as provided by the machine; the machine solution itself is given in Figure 23.

TABLE IV

Selected Points on Solution to Equation 4, as Calculated by Use of Series, and as Provided by Computer

T	β, Calculated	β, from Machine Solution
0	0.0000	0.00
0.4	0.0799	0.080
0.8	0.3157	0.316
1.2	0.6753	0.671
1.6	1.0200	1.020
1.890	1.1268	1.130

Figure 22 shows the set-up used. The general scheme will be obvious, but it should be explained that in actual practice it was found to be more convenient to solve equation 2 with a = 4/3 and with b = 80 than to solve Equation 5 as it stands; the scaling was a bit simpler then. By virtue of the substitutions mentioned above, it clearly makes no difference what values of a and b are used in the machine set-up.

It has already been said that there is little instructional value in seeing how the machine was used in the solution of this problem, but that interest in the problem lies rather in the fact that it was useful for a good accuracy test. However, there exists one feature in Figure 22 that is instructive and this will be commented upon.

If one pretends he does not know what the equation is for which Figure 22 is set-up, and traces out the circuit to deduce that equation, he will not arrive at Equation 5, but rather will find the first term, unity, missing. This term is not produced by any part of the machine during solution and so does not appear in the set-up diagram. This situation should strike the reader as puzzling, if he has not met such a problem before, and indeed even when the answer is known it requires some

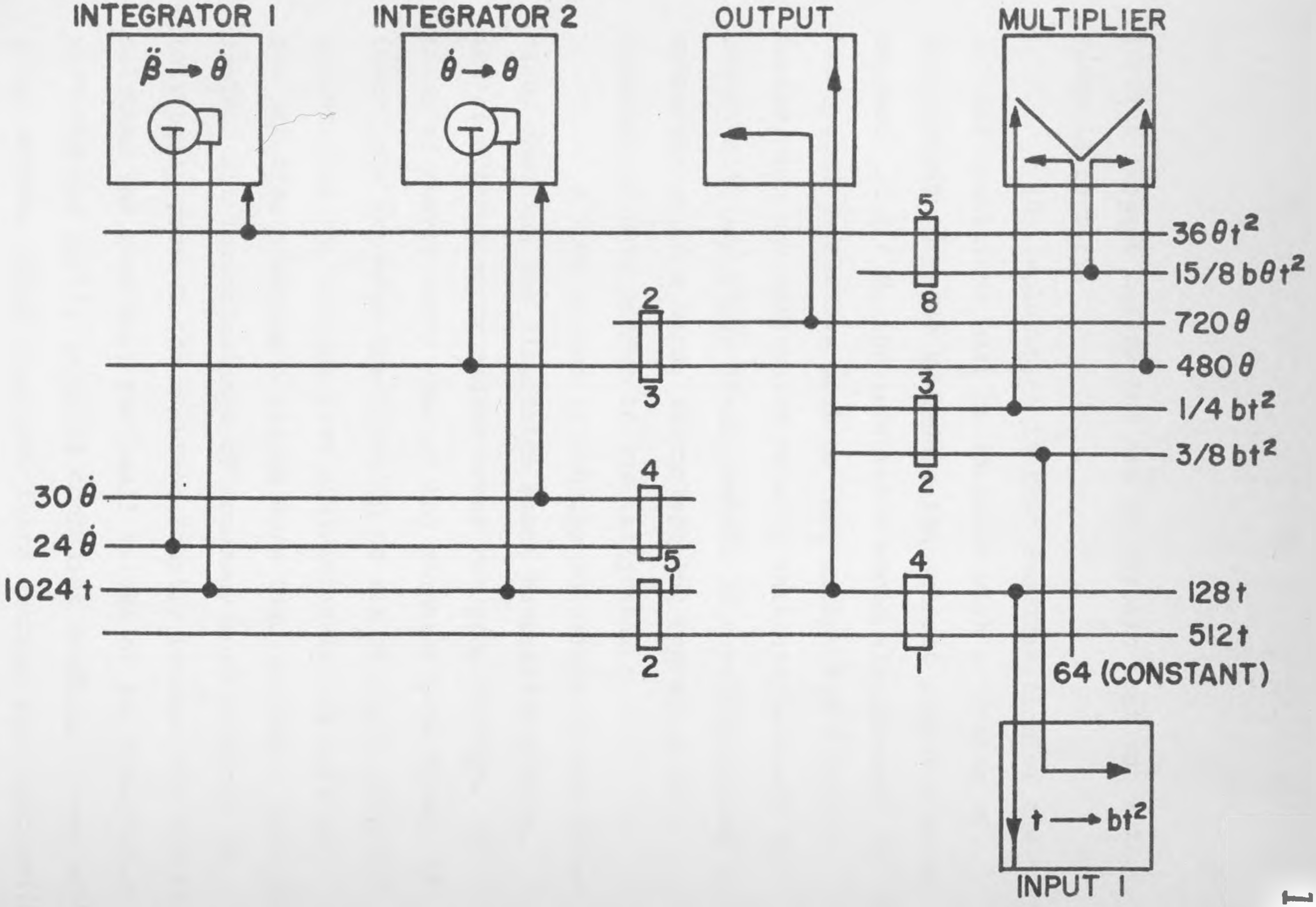
INTEGRATOR I
$\ddot{\beta} \rightarrow \dot{\theta}$
INTEGRATOR 2
$\dot{\theta} \rightarrow \theta$
OUTPUT
MULTIPLIER
$36\theta t^2$
$15/8\, b\theta t^2$
720θ
480θ
$1/4\, bt^2$
$3/8\, bt^2$
$30\dot{\theta}$
$24\dot{\theta}$
$1024t$
$128t$
$512t$
64 (CONSTANT)
$t \rightarrow bt^2$
INPUT I
5
8
2
3
3
2
4
5
2
4
1

FIG. 22.

careful thought before one can be certain that the set-up shown is correct.

The explanation is that consideration of the initial conditions must be included with a tracing of the circuit in order to deduce the correct equation being solved. At T = 0, the integrator whose displacement is the sedond derivative must be one, except for a scale factor, and the integrator retains this displacement in addition to any other displacements it receives during operation -- in a word, the integrator remembers the presence of this number in its integrand.

A long period of running solutions of the equation, checking the dispersion among successive curves, and of attempting to reduce error was gone through. A total of nearly forty runs of the computer were made. Of these, six extended the solution to six or seven complete cycles, and the balance were restricted to the part of the solution covering a little more than the first quarter cycle. All investigations of accuracy were confined to this early part of the solution, partly because the series solution was good only for small values of the independent variable and partly to avoid confusing backlash errors with other errors. From this work there emerged some information about the accuracy of the machine's performance, and certain

other findings of interest for future applications of the machine. In the interests of brevity only the main results will be given.

The first curves drawn showed maximum deviation from their mean of nearly 5%, and correspondingly large failure of the mean to agree with the known solution. The difficulty was found to be that the zeroization of one integrator had become faulty; this illustrates the need for frequent checking of the various parts of the machine. After this trouble had been corrected, the curves produced had a mean that agreed with the known solution to about 0.4%, but the variance from the mean was still as large as 2% for some curves.

It was thought that errors in setting the initial conditions were the largest contributing factors, and to verify this, a new machine set-up was designed and installed, which allowed the machine to solve the early part of the problem on a scale that was more advantageous. In particular, while one integrator setting remained zero; the other one became 20 turns where it had been 4 turns before. As so often happens in testing the machine, somewhat puzzling results were gotten, with the agreement between the mean curves and the known solution being better, but with the dispersion being a little worse.

It may be that the error in setting the zero displacement was still relatively large enough to account for this effect.

In any event, the dependence of the solution on errors in the settings was shown by beginning the solution at a point about one third of the way to the peak. This had the effect of making the setting that had been zero become of the order of ten; the other setting was similarly large. The change in the dispersion of the curves was startling; they showed a mean dispersion of about 0.4%. However, this must be regarded as partly accidental, for one could scarcely expect a figure smaller than 0.8% to arise, on the basis of previous experience.

The original machine set-up was then reinstalled, and solutions were run again, with extreme care being exercized in setting the integrators. The collars serving to prevent end-to-end motion of the lead screw were drawn against the plates supporting the screw with considerable tightness, the zeros were checked, and the displacement screw handle was turned in such a direction during the setting that when the machine was turned on, there would be no play between carriage and screw to be taken up before motion occurred.

The resulting curves showed a mean deviation from their mean of about 0.8%, and maximum variance for any one curve from the mean was about 1.1%, and the mean curve itself agreed with the correct peak value to perhaps 0.2%, although it was difficult to determine this small quantity well. After a week spent investigating other matters, a new set of four solutions was run off, with the same results, except that the maximum variance was slightly less.

Finally the curve of which a tracing is given in Figure 23 was gotten by starting the solution three times, each time observing how closely the solution being produced came to the calculated points. On the third trial, the solution was about as close to the calculated points as were the means of the many runs previously made, and this solution was continued out for the distance seen in the figure. Near the first peak are shown dotted lines that are meant to indicate the region within which any machine solution would be expected to lie if the settings of the initial conditions are carefully done; the size of this region is meant to correspond to the maximum deviation of 1.1%. The error for successive peaks can be taken to be about 1-1/2% per peak, cumulatively.

From the investigation of this problem it was learned that even when the initial conditions affect

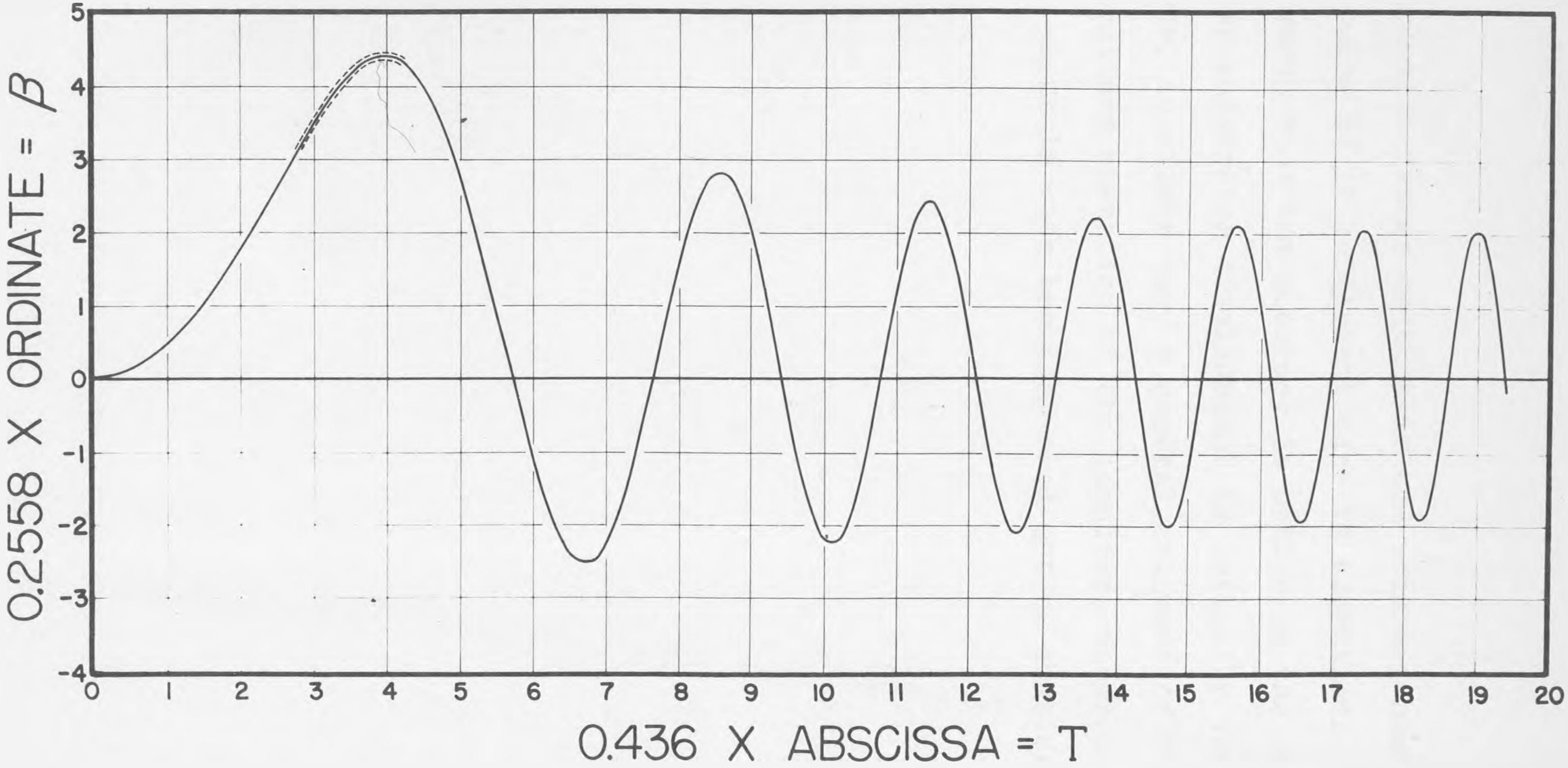
0.2558 X ORDINATE = β
5
4
3
2
1
0
-1
-2
-3
-4
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
0.436 X ABSCISSA = T

FIG. 23.

the solution rather critically the accuracy need not fall much below 1% if sufficient care is exercized. It was also shown that the accuracy is better on the whole when neither integrator displacement is initially very small. However, in either case a general estimate of the accuracy, for ordinary operation of the computer, seems best taken to be about 1% when backlash need not be considered.

Chapter VI

SUMMARY AND REMARKS ON USE OF THE MACHINE

(1) Care of the Machine

It seems advisable to complete the discussion of the State University of Iowa computer by collecting together here some suggestions concerning maintenance of the machine, by summarizing the situation with regard to accuracy attainable, by pointing out what constitute the best applications for it, and, since this is a question likely to be of considerable interest, to mention the use of the machine in the solution of partial differential equations, and in other kinds of problems.

The machine is quite free from danger of any serious breakdowns, and has an advantage over electronic analog computers, and over digital computers, that it need have little idle time for repairs. Since so much of the machine is of aluminum, even rusting is not a problem, provided the shafts are kept coated with a film of oil.

Some things will occur spontaneously to the operator as being worthy of occasional inspection; some of these are that the motor belts should be kept taut, those gears that are permanent as distinguished from those

that are temporarily in place in a set-up, should be checked quite often to be sure they are not slipping on their shafts, the seating of the permanent shafts everywhere should be checked to see that settling or warping of parts has not caused binding or drag to be exerted on them, and the chain drives must not be allowed to become too loose.

Also, the operator will realize the extreme importance of watching the behavior of the integrators carefully almost continually. The zeroizations and the end play of the lead screws especially affect accuracy vitally. The torque amplifiers also must be watched, but these usually give unmistakable warning when they need attention.

When the machine is being run, it will sometimes happen that a gear somewhere in the shaft system will loosen on its shaft. This is a nuisance of sufficient magnitude to make it worth while to go quickly over the shaft system before each run with a wrench, testing each gear in turn and tightening any set screws fround to be loose.

Setting such rules as these is not mere fussiness; the computer is already quite free of troubles whether idle or operating, and if those things mentioned

above that have been found to occur occasionally are prevented by regular inspection and attention, the computer will be found to be nearly completely trouble free.

(2) Accuracy

Perhaps the best way to summarize what the accuracy of the machine is would be the following. If the problem being solved involves no large amounts of backlash during its solution, and if one solution is run, it is safe to expect that the curve obtained will lie not more than 1.2% from the true answer; if a set of four or more solutions are made, the average curve can be safely expected to lie not more than 0.8% from the true answer. If backlash errors are to be expected, these two figures each become 2% and each time the backlash effects occur, the error increases by that amount. If frontlash introduction is used, the figure becomes 1.5%.

To these statements must be added the remark that all these figures are very safely large ones, and can be materially improved on. In favorable cases and with care, a set of successive runs can be made so consistently that the pencil line drawn in one run becomes only slightly broadened through superposition of the subsequent runs, and all will agree with the true answer by a comparable

amount. It should always be borne in mind that pushing the machine to the upper limit of its accuracy is a delicate matter and cannot always be done.

It must also be realized that if one is dealing with a problem with no means available for checking the accuracy being gotten, then one's confidence in the accuracy must not be allowed to rise too high. It would be better to make a practice of using the safe figures given above, and, if it is desired to be able to guarantee better precision, to construct a similar equation and observe the accuracy one gets in solving it, in the manner outlined earlier in this paper.

(3) Economic Use of the Machine

When problems exist that cannot be treated in any way other than by use of the computer, recourse will be had to it in any event, but sometimes the question may arise, whether it is worth while to employ the machine. For example, the machine could be used to perform a simple integration of a known function, but if one can do the same thing by numerical methods in one or two afternoons of work, it might be better to do so than to use the machine.

All in all, the time required to work out a set-up

diagram, including any exploratory work that may be needed to find a suitable form for the problem or to discover the ranges of the variables, the time required to put the shafts and gears into the interconnecting system, and, by no means least, the time required in making and supporting an estimate of the accuracy being gotten, are the factors that will decide the question. If an equation has been dealt with before or is especially simple, a few hours will suffice for all this work, but as a general rule one can expect to be occupied with a given problem for more than a week, at a minimum.

If only one solution of an equation is wanted and if high accuracy is essential, there will be no saving of time by use of the machine. However, if a number of solutions of the same equation are wanted, or if the problem is one of investigating the effect of variation of a parameter on the solution, or if one has a "jury" problem (mentioned previously), or in any of a number of other instances that will occur to the reader, the preliminary work of preparing the machine will be negligible compared to the saving of time that would otherwise be spent in numerical calculations. If the problem is extensive enough to make several weeks of use of one set-up possible, then use of the machine would be

very economical of time.

It seems appropriate to add here that since a number of differential analyzers have been in existence for many years, and since a great deal of work has been done with them and reported in the literature, there is no difficulty in finding accounts of treatments of problems in the most diverse fields. Much can be learned from examination of some of these accounts, and if one can find a record of treatment of a problem similar to one he is faced with, some time and work may be saved.

(4) Other Applications

The differential analyzer is, by its nature, especially adapted to the solution of ordinary differential equations, and the State University of Iowa machine in particular is limited at present to second order equations in one variable. However, extension of the utility of the machine can be accomplished rather easily, and too, it has other uses as it stands.

In the first place, ingenuity alone is required to see whether any given problem can be treated with the computer, and it is not necessary to attempt to list possible applications here. Wherever research programs are being conducted, the need for evaluation of integrals

will frequently arise, and only two or three afternoons of work would be required to connect an input table, integrator, and output table to perform such integrations, so that it might be expected that the machine will be of use in such cases.

In the second place, increase of the order of the equations that can be handled could be achieved by building more integrators. With more integrators available, other possibilities begin to arise for applications, too. For one example, systems of equation of second order in more than one variable then could be solved, such as three dimensional rocket problems.

Again, there exist many possibilities of using parts of the machine in the treatment of problems of an entirely different sort. These problems need not even involve the use of the calculus, and as one example it might be mentioned that purely algebraic problems have been treated with a set of interconnected input units.

There is, in conclusion, one question of very great interest; that question is whether such a machine can be used to solve partial differential equations. The answer is yes, but to explain how it can be done requires lengthy discussion. For the sake of any readers

who wish to pursue this matter, the writer recommends the work by Crank[22], in which much other material of utility will also be found.

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